The Number of Determining Modes in 2D Turbulence: A Computational Study

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The method of continuous data assimilation from weather forecasting is used to study the number of determining modes for the two-dimensional incompressible Navier–Stokes equations. Our focus is on how the body forcing affects the the rate of continuous data assimilation and the number of determining modes. These quantities are shown to depend on the length scales present in the forcing.

Outline of Talk

- Continuous Data Assimilation
- The 2D Navier–Stokes Equations
- Determining Modes
- Analytical Results
- Computations

Continuous Data Assimilation

In the late 1960s satellite-borne observation systems began producing data on the climate that was nearly continuous in time.

Charney, Halem and Jastrow (1969) proposed that the equations of the atmosphere themselves be used to process this data and obtain improved estimates of the current atmospheric state.

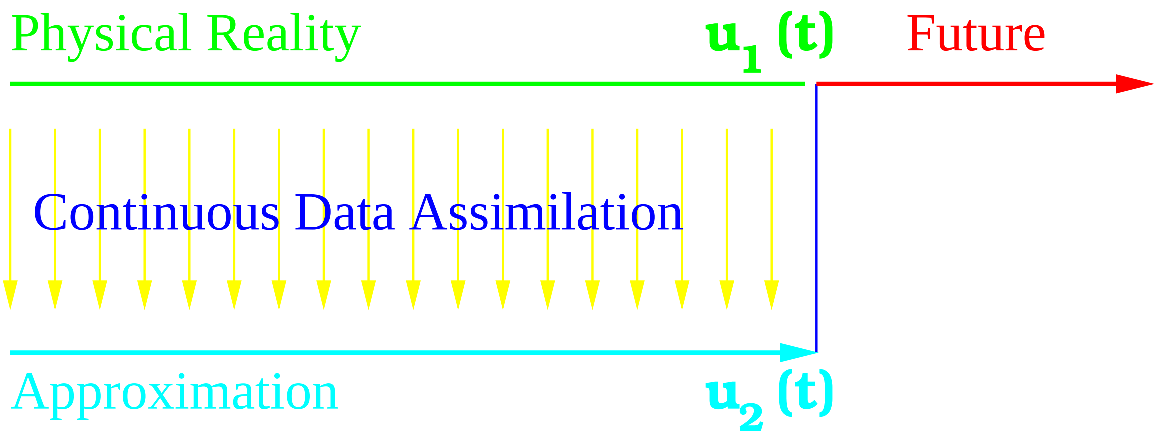
Their method, called continuous data assimilation, is in principle, straightforward:

> Data is inserted directly into a numerical model
> as it is being integrated in time.

An improved estimate of the atmospheric state is thus obtained and can serve as initial conditions for subsequent prediction.

Continuous Data Assimilation

The observational measurments of the physical reality are inserted directly into the numerical model at each time step.



Continuous Data Assimilation

We study continuous data assimilation for the two-dimensional Navier–Stokes equations.

Our work is

- Motivated by applications involving the full dynamics of the atmosphere.

- Inspired by the work of Browning, Henshaw and Kreiss.

Continuous Data Assimilation

We work on the 2π-periodic torus Ω.

> Fourier space provides a convenient way of representing the observational measurements needed for our study.

Let the observational measurement of u_1 at time t be given by the projection $P_\lambda u_1(t)$ where

$$P_\lambda u = \sum_{|k|^2 \le \lambda} \hat{u}_k e^{ik\cdot x} \qquad \text{and} \qquad u = \sum_{k \in \mathbb{Z}^2} \hat{u}_k e^{ik\cdot x}.$$

Note that

- The observational measurements are error free,
- $\lambda^{-1/2}$ is the smallest length scale which can be observed,
- λ and the rank of P_λ are proportional.

Continuous Data Assimilation

Browning, Henshaw and Kreiss (1998) studied continuous data assimilation for the two-dimensional Navier–Stokes equations.

Computations were done with viscosity $\nu = 10^{-5}$ and forcing $f = 0$ starting from an initial condition u_0 such that $|u_0| = 1$.

It was observed that continuous data assimilation of the 64 lowest Fourier modes was sufficient to accurately reconstruct the small scales, but assimilation of the 16 lowest modes was not.

Continuous Data Assimilation

We generalize the work of Browning, Henshaw and Kreiss by considering a non-zero body forcing f.

This avoids the difficulties of u_1 and u_2 both decaying to zero.

Our main focus is

> How do the the spatial length scales present in f affect the convergence of u_2 to u_1?

The 2D Navier–Stokes Equations

The incompressible two-dimensional Navier–Stokes equations are

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u + \nabla \pi = f, \\ \hfill \nabla \cdot u = 0. \end{cases}$$

Here

- u is the Eulerian velocity field,
- ν is the kinematic viscosity,
- π is the physical pressure, and
- f is the body forcing.

The 2D Navier–Stokes Equations

Define

$$H = \left\{ u \in \left[L^2_{\text{per}}(\Omega)\right]^2 : \qquad \nabla \cdot u = 0 \quad \text{and} \quad \int_\Omega u = 0 \right\}$$

and

$$V = H \cap \left[H^1(\Omega)\right]^2.$$

For $u_0 \in V$ and $f \in H$, the 2DNS equations can be written as an evolution equation in H as

$$\frac{du}{dt} + \nu A u + B(u,u) = f.$$

Here

- $Au = -P_H \Delta u$ is the Stokes operator,
- $B(u,u)$ is the non-linear term $P_H(u \cdot \nabla)u$.

The 2D Navier–Stokes Equations

Split the high and low modes of the solution u as

$$p = P_\lambda u \qquad \text{and} \qquad q = Q_\lambda u = (I - P_\lambda)u.$$

Then the 2DNS equations may be written as

$$\begin{cases} \dfrac{dp}{dt} + \nu Ap + P_\lambda B(p+q, p+q) = P_\lambda f, \\ \\ \dfrac{dq}{dt} + \nu Aq + Q_\lambda B(p+q, p+q) = Q_\lambda f. \end{cases}$$

Continuous data assimilation is effected by replacing the equation for p above by the observational measurements.

The 2D Navier–Stokes Equations

Let

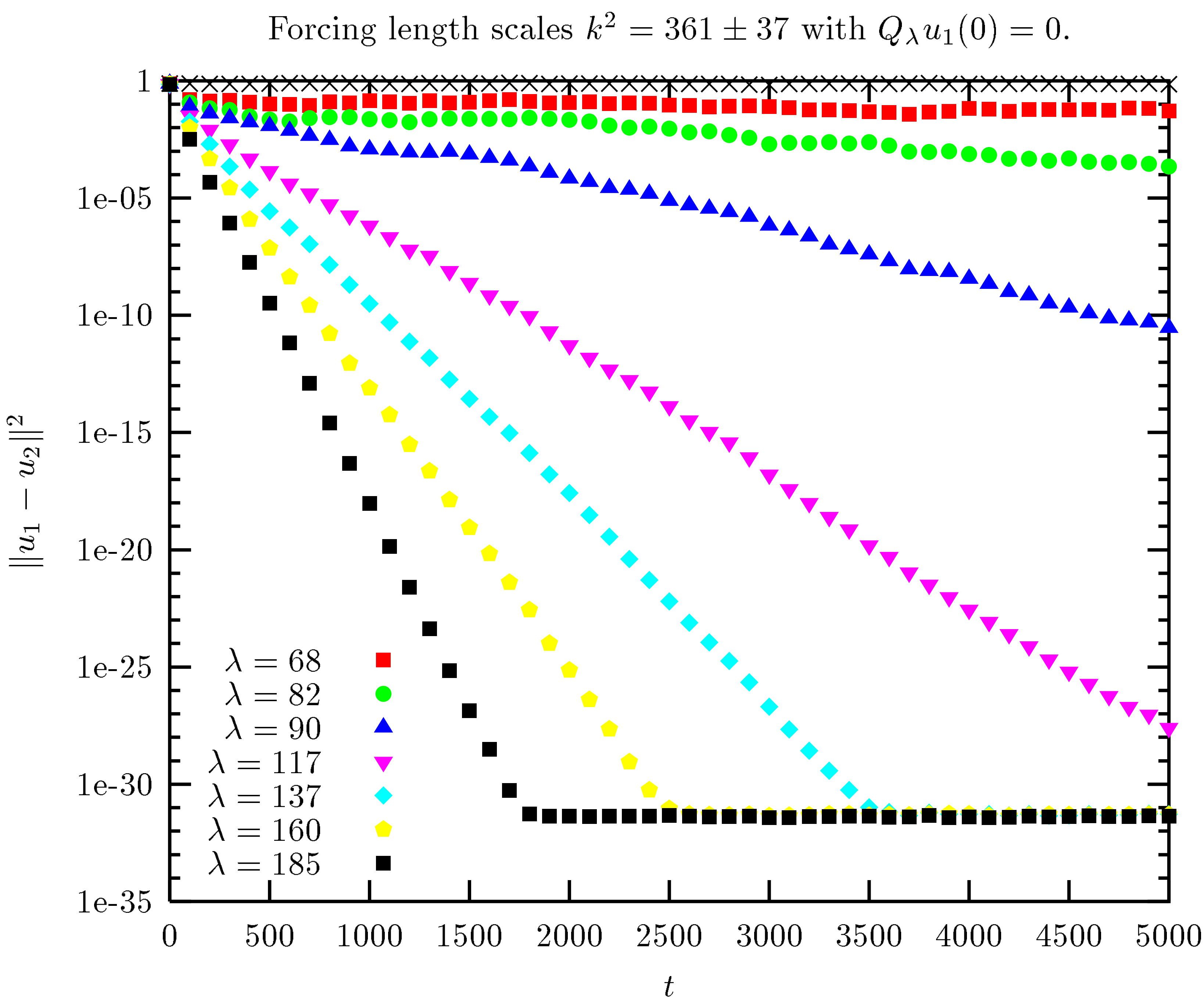
- $u_1 = p_1 + q_1$ be physical reality,
- $u_2 = p_2 + q_2$ be our approximation of reality, and
- $P_\lambda u_1 = p_1$ be the observational measurements.

We compute u_2 via the modified Navier–Stokes equations

$$\begin{cases} p_2 = p_1, \\[2ex] \dfrac{dq_2}{dt} + \nu A q_2 + Q_\lambda B(p_1 + q_2, p_1 + q_2) = Q_\lambda f. \end{cases}$$

It is of practical importance to answer the question,

When will u_2 be a good approximation of u_1?



Forcing length scales $k^2 = 361 \pm 37$ with $Q_\lambda u_1(0) = 0$.
$\|u_1 - u_2\|^2$
1
1e-05
1e-10
1e-15
1e-20
1e-25
1e-30
1e-35
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
t
$\lambda = 68$
$\lambda = 82$
$\lambda = 90$
$\lambda = 117$
$\lambda = 137$
$\lambda = 160$
$\lambda = 185$

Determining Modes

It was shown by Foias and Prodi (1967) that the 2DNS equations have a finite number of determining modes.

The number of determining modes for the 2DNS equations is the rank of the smallest projection P_λ such that

$$|P_\lambda u_1 - P_\lambda u_2| \to 0 \qquad \text{as} \qquad t \to \infty$$

guarantees that

$$\|u_1 - u_2\| \to 0 \qquad \text{as} \qquad t \to \infty$$

for any two solutions u_1 and u_2 of the 2DNS equations

$$\frac{du}{dt} + vAu + B(u,u) = f.$$

Determining Modes

Trève and Manley (1981) gave a physical argument relating the number of determining modes in Rayleigh-Bénard convection to the Rayleigh number divided by the Prandtl number.

By identifying the buoyancy force in Rayleigh-Bénard convection with the body forcing in the Navier-Stokes equations this lead to

the Grashof number defined as

$$\mathrm{Gr}(f) = \nu^{-2} \limsup_{t\to\infty} \big| f(t) \big|.$$

The first reasonable estimate on the number of determining modes in terms of the Grashof number was provided by Foias, Manley, Temam and Trève (1983).

Determining Modes

Jones and Titi (1993) provide the best estimate to date.

Theorem. Let v_i be strong solutions to the 2DNS equations

$$\frac{dv_i}{dt} + \nu A v_i + B(v_i, v_i) = f_i$$

for $i = 1, 2$. If $\lambda > c_1 \mathrm{Gr}(f_1)$, then

$$|f_1 - f_2| \to 0 \qquad \text{and} \qquad |P_\lambda v_1 - P_\lambda v_2| \to 0$$

as $t \to \infty$ implies $\|v_1 - v_2\| \to 0$ as $t \to \infty$.

This theorem also relaxes the assumption that v_1 and v_2 both be solutions of the same dynamical system by allowing the forcing functions to be different for each solution.

Determining Modes

Since the difference of the low modes of v_1 and v_2 are controlled by hypothesis, only the difference of the high modes of f_1 and f_2 actually enter into the proof of the preceding result.

This observation yields the following theorem:

Theorem 1. Let v_i be strong solutions to the 2DNS equations

$$\frac{dv_i}{dt} + \nu A v_i + B(v_i, v_i) = f_i$$

for $i = 1, 2$. If $\lambda > c_1 \mathrm{Gr}(f_1)$, then

$$|Q_\lambda f_1 - Q_\lambda f_2| \to 0 \qquad \text{and} \qquad |P_\lambda v_1 - P_\lambda v_2| \to 0$$

as $t \to \infty$ implies $\|v_1 - v_2\| \to 0$ as $t \to \infty$.

We now rewrite the continuous data assimilation equations in such a way that Theorem 1 applies.

Determining Modes

Add the evolution equation for p_1 to the evolution equation for q_2.

$$\frac{dp_1}{dt} + \nu A p_1 + P_\lambda B(u_1, u_1) = P_\lambda f$$

and

$$\frac{dq_2}{dt} + \nu A q_2 + Q_\lambda B(u_2, u_2) = Q_\lambda f$$

yields

$$\frac{du_2}{dt} + \nu A u_2 + B(u_2, u_2) = f + P_\lambda\big(B(u_2, u_2) - B(u_1, u_1)\big).$$

Determining Modes

Thus, we obtain the coupled system of equations

$$\begin{cases} \dfrac{du_1}{dt} + \nu A u_1 + B(u_1, u_1) = f, \\ \\ \dfrac{du_2}{dt} + \nu A u_2 + B(u_2, u_2) = f + P_\lambda\big(B(u_2, u_2) - B(u_1, u_1)\big) \end{cases}$$

with initial conditions $u_1(0) = u_0$ and $u_2(0) = P_\lambda u_0 + Q_\lambda \eta$.

Note that

- We take $u_0 \in V$, $\eta \in Q_\lambda V$ and $f \in H$,
- η is an arbitrary initialization of the high modes of u_2,
- We take $\eta = 0$ in our numerical computations.

Determining Modes

Define

$$f_1 = f \qquad \text{and} \qquad f_2 = f + P_\lambda\big(B(u_2, u_2) - B(u_1, u_1)\big).$$

Now consider the solutions v_i to the 2DNS equations

$$\frac{dv_i}{dt} + \nu A v_i + B(v_i, v_i) = f_i \qquad \text{for} \qquad i = 1, 2.$$

Since

$$|Q_\lambda f_1 - Q_\lambda f_2| = 0 \qquad \text{and} \qquad |P_\lambda v_1 - P_\lambda v_2| = 0,$$

Theorem 1 implies that if $\lambda > c_1 \mathrm{Gr}(f)$, then

$$\|v_1 - v_2\| \to 0 \qquad \text{as} \qquad t \to \infty.$$

Analytical Results

Uniqueness of strong solutions implies

$$u_1(t) = v_1(t) \qquad \text{and} \qquad u_2(t) = v_2(t).$$

Therefore,

$$\|u_1 - u_2\| \to 0 \qquad \text{as} \qquad t \to \infty.$$

This would be the end of the story, except

- We need $f_2 \in L^2_{\text{loc}}\big((0,\infty); H\big)$ for unique strong solutions.
- $\text{Gr}(f)$ doesn't depend on the spatial length scales in f.
- Theorem 1 provides no estimate on the rate of convergence.

Analytical Results

The following theorem addresses the first problem.

Theorem 2. If $u_0 \in V$, $\eta \in Q_\lambda V$ and $f \in L^2_{\mathrm{loc}}\big((0,\infty);H\big)$ then the continuous data assimilation equations viewed as a system of functional equations in H has a unique strong solution that satisfies

$$u_i \in L^\infty\big((0,T);V\big) \cap L^2\big((0,T);D(A)\big) \quad \text{and} \quad \frac{du_i}{dt} \in L^2\big((0,T);H\big)$$

for $i = 1,2$ and any $T > 0$. Furthermore, the solutions are in $C\big([0,T];V\big)$ and depend continuously on the initial data u_0 and η in the V norm.

Analytical Results

To address the second problem we employ the norm on the dual space V^* of V defined by

$$\|u\|_* = \sup\left\{ \langle u, v\rangle : \quad v \in V \quad \text{and} \quad \|v\| = 1 \right\}.$$

Expressing u in terms of the formal Fourier series

$$u = \sum_{k\in\mathbf{Z}^2} \hat{u}_k e^{ik\cdot x} \qquad \text{where} \qquad \hat{u}_k = \overline{\hat{u}_{-k}}, \quad k\cdot\hat{u}_k = 0, \quad \hat{u}_0 = 0$$

the spaces H, V and V^* may be expressed as

$$H = \left\{ u : |u| < \infty \right\} \qquad \text{where} \qquad |u|^2 = 4\pi^2 \sum \left|\hat{u}_k\right|^2,$$

$$V = \left\{ u : \|u\| < \infty \right\} \qquad \text{where} \qquad \|u\|^2 = 4\pi^2 \sum |k|^2 \left|\hat{u}_k\right|^2,$$

$$V^* = \left\{ u : \|u\|_* < \infty \right\} \qquad \text{where} \qquad \|u\|_*^2 = 4\pi^2 \sum |k|^{-2} \left|\hat{u}_k\right|^2.$$

Analytical Results

Our main result is

Theorem 3. Let $u_1(t)$ be a solution on the global attractor of the two-dimensional Navier–Stokes equations with forcing $f \in H$. Let $u_2(t)$ be the approximation to $u_1(t)$ obtained from the continuous data assimilation of the observational measurements $P_\lambda u_1(\tau)$ over the time interval $\tau \in [0, t]$. Then there are constants K_1 and K_2 independent of all initial conditions such that

(i) If there exists α such that $0 < 2\alpha \leq \nu\lambda - c_1^2 \nu^{-3} \|f\|_*^2$ then

$$\big|u_1(t) - u_2(t)\big| \leq \big|u_1(0) - u_2(0)\big| K_1 e^{-\alpha t} \qquad \text{for} \qquad t \geq 0.$$

(ii) If there exists α such that $0 < 2\alpha \leq \nu\lambda - c_1^2 (\nu^3 \lambda)^{-1} |f|$ then

$$\big\|u_1(t) - u_2(t)\big\| \leq \big\|u_1(0) - u_2(0)\big\| K_2 e^{-\alpha t} \qquad \text{for} \qquad t \geq 0.$$

Analytical Results

Sketch of proof: Let $\delta = u_1 - u_2$. Since u_1 and u_2 agree on the Fourier modes with $|k|^2 \le \lambda$, then

$$|\delta|^2 = 4\pi^2 \sum_{|k|^2 > \lambda} \left|\hat{\delta}_k\right|^2 \le \frac{4\pi^2}{\lambda} \sum_{|k|^2 > \lambda} |k|^2 \left|\hat{\delta}_k\right|^2 = \lambda^{-1} \|\delta\|^2.$$

From this improved Poincaré inequality we obtain

$$\begin{aligned} \left|\delta(t)\right|^2 &\le \left|\delta(0)\right|^2 \exp\left\{ -\nu\lambda t + \frac{c_1^2}{\nu} \int_0^t \left\|u_1(\tau)\right\|^2 d\tau \right\} \\ &\le \left|\delta(0)\right|^2 \exp\left\{ \frac{c_1^2}{\nu^4} |f|_*^2 \right\} \exp\left\{ \left(-\nu\lambda + \frac{c_1^2}{\nu^3} |f|_*^2 \right) t \right\} \end{aligned}$$

and

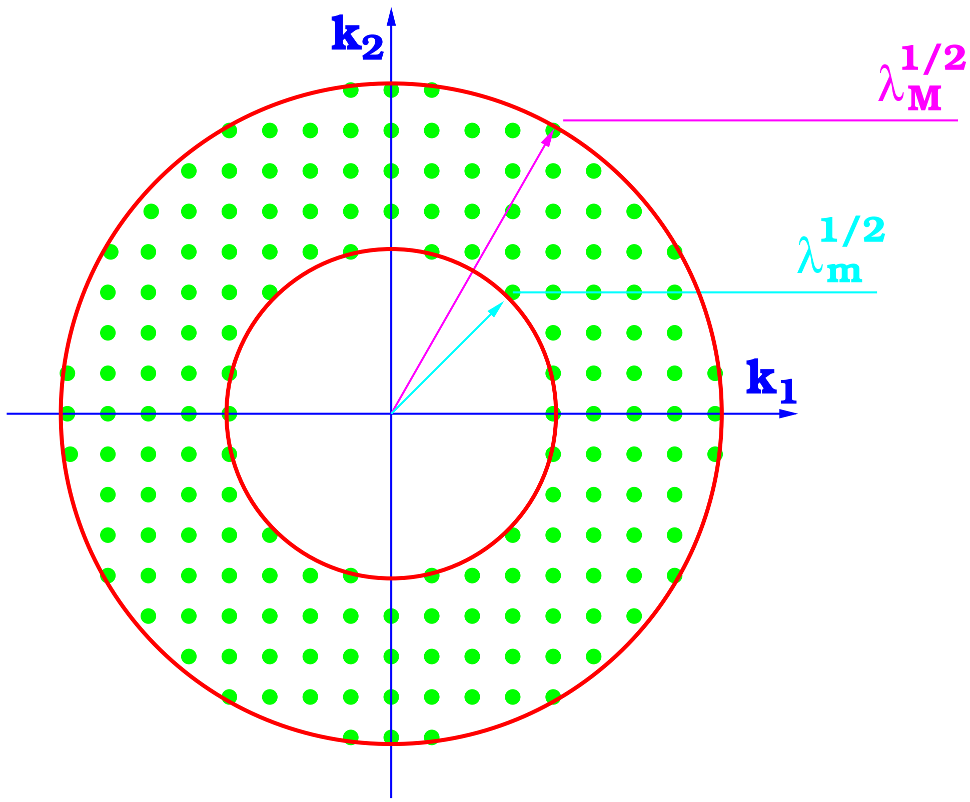
$$\begin{aligned} \left\|\delta(t)\right\|^2 &\le \left\|\delta(0)\right\|^2 \exp\left\{ -\nu\lambda t + \frac{c_1^2}{\nu\lambda} \int_0^t \left|Au_1(\tau)\right|^2 d\tau \right\} \\ &\le \left\|\delta(0)\right\|^2 \exp\left\{ \frac{c_1^2}{\nu^4\lambda} |f|^2 \right\} \exp\left\{ \left(-\nu\lambda + \frac{c_1^2}{\nu^3\lambda} |f|^2 \right) t \right\}. \end{aligned}$$

Computations

Consider $f \in H$ with $\mathrm{Gr}(f) = 250000$ that are supported on an annulus in Fourier space. Thus f may be written

$$f = \sum_{\lambda_m \leq |k|^2 \leq \lambda_M} \hat{f}_k e^{ik\cdot x}.$$

Graphically



Computations

Note that

$$|f|_*^2 = \sum_{\lambda_m \le |k|^2 \le \lambda_M} |k|^{-2} \left|\hat{f}_k\right|^2 \le \frac{1}{\lambda_m} |f|^2.$$

Therefore $|f|_*$ decreases as λ_m increases.

The upper bound on the number of determining modes given in Theorem 3 therefore decreases as λ_m increases.

In particular, if $c_1^2 \nu^{-4} \|f\|_*^2 < 1$, then the upper bound in Theorem 3 becomes trivial and $\|u_1 - u_2\| \to 0$ for any λ.

Computations

As an aside, this line of reasoning obtains the general result:

> Let f be supported only on modes in Fourier space with $|k|^2 \geq c_1^2 \nu^{-4} |f|^2$. Then any solution to the 2DNS equations converges to an asymptotically stable steady flow.

This contrasts to the (1986) result of Marchioro:

> If f is supported on only the lowest modes in Fourier space, then any solution to the 2DNS equations converges to an asymptotically stable steady flow.

Computations

For our first set of computations we set the width of the annulus

$$\lambda_M - \lambda_m = 4\lambda_f^{1/2} - 2 \qquad \text{where} \qquad \lambda_f = (\lambda_m + \lambda_M)/2.$$

This ensures that f immediately generates interactions involving all Fourier modes through the non-linear term.

Thus, we consider the following annuli

λ_f	λ_m	λ_M
25	16	34
64	49	79
121	100	142
169	144	194
256	255	287
361	324	398
484	441	527
529	484	574
576	529	623
625	576	674

Computations

Forcing functions supported on different length scales, their norms, the critial resolution parameter for the observational measurements and the corresponding minimal number of determining modes.

λ_f	$\|f\|$	$|f|$	$\|f\|_*$	λ_c	N
25	0.01145	0.0025	0.0005588	4	12
64	0.01928	0.0025	0.0003272	13	44
121	0.02674	0.0025	0.0002350	26	88
169	0.03180	0.0025	0.0001974	49	148
256	0.03941	0.0025	0.0001589	73	232
361	0.04717	0.0025	0.0001328	82	260
484	0.05487	0.0025	0.0001141	73	232
529	0.05716	0.0025	0.0001095	65	212
576	0.05954	0.0025	0.0001051	26	88
625	0.06223	0.0025	0.0001005	10	36

Computations

Forcing functions of the form $f = f_L + f_H$ with $|f| = 0.0025$.

- f_L is supported on modes with $16 \leq |k|^2 \leq 34$,
- f_H is supported on modes with $441 \leq |k|^2 \leq 527$.

The critical resolution parameter for the observational measurements and the minimum number of determining modes depends on the relative amount of forcing in the high and low modes.

$\|f\|$	$\vert f_L \vert$	$\vert f_H \vert$	$\|f\|_*$	λ_c	N
0.05487	0	0.002500	0.0001141	73	232
0.05469	0.000200	0.002492	0.0001222	37	120
0.05418	0.000400	0.002468	0.0001438	26	88
0.05210	0.000800	0.002369	0.0002090	13	44
0.03620	0.001921	0.001600	0.0004356	5	20
0.01145	0.002500	0	0.0005588	4	12