Math/CS 467/667 Programming Assignment 1

1. The Lorenz system is a three dimensional ordinary differential equation of the form

$$\frac{dy}{dt} = f(y)$$

with a given initial condition $y(0) = a$ where $y(t)$ is a vector in $\mathbf{R}^3$ and

$$f(y) = \begin{bmatrix} -10y_1 + 10y_2 \\ 28y_1 - y_2 - y_1y_3 \\ y_1y_2 - (8/3)y_3 \end{bmatrix}.$$

Let Y^n be an approximation of $y(1)$ obtained using a step size of $h = 1/n$. Define the error

$$E_n = \|Y^n - y(1)\| = \left\{ \sum_{i=1}^{3} \big(Y_i^n - y_i(1)\big)^2 \right\}^{1/2}.$$

Show that if $E_n \le Kh^k$ then

$$\|Y^n - Y^{2n}\| \le K\left\{1 + \frac{1}{2^k}\right\} h^k.$$

2. Write a program to approximate solutions of the Lorenz system using Euler's forward difference method and the initial condition

$$a = \begin{bmatrix} 2 \\ 3 \\ 15 \end{bmatrix}.$$

Compute Y^n for $n = 64, 128, 256, 512, \ldots, 65536$.

3. Compute Y^n using Runge-Kutta methods of orders 2, 3 and 4 given by the tableux

$$\begin{array}{c|cc} 0 & & \\ \frac{2}{3} & \frac{2}{3} & \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}, \qquad \begin{array}{c|ccc} 0 & & & \\ \frac{2}{3} & \frac{2}{3} & & \\ \frac{2}{3} & 0 & \frac{2}{3} & \\ \hline & \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \end{array} \quad \text{and} \quad \begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array},$$

respectively, and verify the order by graphing $\log \|Y^n - Y^{2n}\|$ versus $\log h$.

4. Approximate $y(10)$ to three decimal places. Is it possible to achieve this accuracy using Euler's method? Can you find $y(100)$?