```
> restart;
> # Compute truncation error for theta method
> eq[1] := (D[2](U))(x, t+(1/2)*dt) = (D[1, 1](U))(x, t+(1/2)*dt);
```

$$eq_1 := \mathrm{D}_2(U)\left(x, t + \frac{1}{2}\,dt\right) = \mathrm{D}_{1,1}(U)\left(x, t + \frac{1}{2}\,dt\right)$$

```
> eq[2] := (D(unapply(eq[1], x)))(x);
```

$$eq_2 := \mathrm{D}_{1,2}(U)\left(x, t + \frac{1}{2}\,dt\right) = \mathrm{D}_{1,1,1}(U)\left(x, t + \frac{1}{2}\,dt\right)$$

```
> eq[3] := (D(unapply(eq[1], t)))(t);
```

$$eq_3 := \mathrm{D}_{2,2}(U)\left(x, t + \frac{1}{2}\,dt\right) = \mathrm{D}_{1,1,2}(U)\left(x, t + \frac{1}{2}\,dt\right)$$

```
> eq[4] := (D(unapply(eq[2], x)))(x);
```

$$eq_4 := \mathrm{D}_{1,1,2}(U)\left(x, t + \frac{1}{2}\,dt\right) = \mathrm{D}_{1,1,1,1}(U)\left(x, t + \frac{1}{2}\,dt\right)$$

```
> eq[5] := (D(unapply(eq[3], t)))(t);
```

$$eq_5 := \mathrm{D}_{2,2,2}(U)\left(x, t + \frac{1}{2}\,dt\right) = \mathrm{D}_{1,1,2,2}(U)\left(x, t + \frac{1}{2}\,dt\right)$$

```
> eq[6] := (D(unapply(eq[3], x)))(x);
```

$$eq_6 := \mathrm{D}_{1,2,2}(U)\left(x, t + \frac{1}{2}\,dt\right) = \mathrm{D}_{1,1,1,2}(U)\left(x, t + \frac{1}{2}\,dt\right)$$

```
> eq[7] := (D(unapply(eq[4], t)))(t);
```

$$eq_7 := \mathrm{D}_{1,1,2,2}(U)\left(x, t + \frac{1}{2}\,dt\right) = \mathrm{D}_{1,1,1,1,2}(U)\left(x, t + \frac{1}{2}\,dt\right)$$

```
> eq[8] := (D(unapply(eq[4], x)))(x);
```

$$eq_8 := \mathrm{D}_{1,1,1,2}(U)\left(x, t + \frac{1}{2}\,dt\right) = \mathrm{D}_{1,1,1,1,1}(U)\left(x, t + \frac{1}{2}\,dt\right)$$

```
> eq[9] := (D(unapply(eq[8], x)))(x);
```

$$eq_9 := \mathrm{D}_{1,1,1,1,2}(U)\left(x, t + \frac{1}{2}\,dt\right) = \mathrm{D}_{1,1,1,1,1,1}(U)\left(x, t + \frac{1}{2}\,dt\right)$$

```
> eq[10] := (D(unapply(eq[9], t)))(t);
```

$$eq_{10} := \mathrm{D}_{1,1,1,1,2,2}(U)\left(x, t + \frac{1}{2}\,dt\right) = \mathrm{D}_{1,1,1,1,1,1,2}(U)\left(x, t + \frac{1}{2}\,dt\right)$$

```
> eq[11] := (D(unapply(eq[9], x)))(x);
```

$$eq_{11} := \mathrm{D}_{1,1,1,1,1,2}(U)\left(x, t+\frac{1}{2}dt\right) = \mathrm{D}_{1,1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)$$

```
> eq[12] := (D(unapply(eq[11], x)))(x);
```

$$eq_{12} := \mathrm{D}_{1,1,1,1,1,1,2}(U)\left(x, t+\frac{1}{2}dt\right) = \mathrm{D}_{1,1,1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)$$

```
> A1 := series(U(x, t+dt), dt = (1/2)*h, 4);
```

$$A1 := U\left(x, t+\frac{1}{2}h\right) + \mathrm{D}_2(U)\left(x, t+\frac{1}{2}h\right)\left(dt-\frac{1}{2}h\right) + \frac{1}{2}\mathrm{D}_{2,2}(U)\left(x, t+\frac{1}{2}h\right)\left(dt-\frac{1}{2}h\right)^2 + \frac{1}{6}\mathrm{D}_{2,2,2}(U)\left(x, t+\frac{1}{2}h\right)\left(dt-\frac{1}{2}h\right)^3 + O\left(\left(dt-\frac{1}{2}h\right)^4\right)$$

```
> A2:=convert(subs(h=dt,A1),polynom);
```

$$A2 := U\left(x, t+\frac{1}{2}dt\right) + \frac{1}{2}\mathrm{D}_2(U)\left(x, t+\frac{1}{2}dt\right)dt + \frac{1}{8}\mathrm{D}_{2,2}(U)\left(x, t+\frac{1}{2}dt\right)dt^2 + \frac{1}{48}\mathrm{D}_{2,2,2}(U)\left(x, t+\frac{1}{2}dt\right)dt^3$$

```
> A3 := subs({eq[1], eq[3], eq[5]}, A2);
```

$$A3 := U\left(x, t+\frac{1}{2}dt\right) + \frac{1}{2}\mathrm{D}_{1,1}(U)\left(x, t+\frac{1}{2}dt\right)dt + \frac{1}{8}\mathrm{D}_{1,1,2}(U)\left(x, t+\frac{1}{2}dt\right)dt^2 + \frac{1}{48}\mathrm{D}_{1,1,2,2}(U)\left(x, t+\frac{1}{2}dt\right)dt^3$$

```
> A4 := subs({eq[4], eq[7]}, A3);
```

$$A4 := U\left(x, t+\frac{1}{2}dt\right) + \frac{1}{2}\mathrm{D}_{1,1}(U)\left(x, t+\frac{1}{2}dt\right)dt + \frac{1}{8}\mathrm{D}_{1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dt^2 + \frac{1}{48}\mathrm{D}_{1,1,1,1,2}(U)\left(x, t+\frac{1}{2}dt\right)dt^3$$

```
> A5 := subs(eq[9], A4);
```

$$A5 := U\left(x, t+\frac{1}{2}dt\right) + \frac{1}{2}\mathrm{D}_{1,1}(U)\left(x, t+\frac{1}{2}dt\right)dt + \frac{1}{8}\mathrm{D}_{1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dt^2 + \frac{1}{48}\mathrm{D}_{1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dt^3$$

```
> A6 := series(U(x, t), t = h, 4);
```

$$A6 := U(x, h) + \mathrm{D}_2(U)(x, h)\,(t - h) + \frac{1}{2}\,\mathrm{D}_{2,2}(U)(x, h)\,(t - h)^2 + \frac{1}{6}\,\mathrm{D}_{2,2,2}(U)(x, h)\,(t - h)^3 + O((t - h)^4)$$

```
> A7 := convert(subs(h = t+(1/2)*dt, A6), polynom);
```

$$A7 := U\left(x, t + \frac{1}{2}\,dt\right) - \frac{1}{2}\,\mathrm{D}_2(U)\left(x, t + \frac{1}{2}\,dt\right)\,dt + \frac{1}{8}\,\mathrm{D}_{2,2}(U)\left(x, t + \frac{1}{2}\,dt\right)\,dt^2 - \frac{1}{48}\,\mathrm{D}_{2,2,2}(U)\left(x, t + \frac{1}{2}\,dt\right)\,dt^3$$

```
> A8 := subs(eq[9], subs({eq[4], eq[7]}, subs({eq[1], eq[3], eq[5]}, A7)));
```

$$A8 := U\left(x, t + \frac{1}{2}\,dt\right) - \frac{1}{2}\,\mathrm{D}_{1,1}(U)\left(x, t + \frac{1}{2}\,dt\right)\,dt + \frac{1}{8}\,\mathrm{D}_{1,1,1,1}(U)\left(x, t + \frac{1}{2}\,dt\right)\,dt^2 - \frac{1}{48}\,\mathrm{D}_{1,1,1,1,1,1}(U)\left(x, t + \frac{1}{2}\,dt\right)\,dt^3$$

```
> Left := simplify((A5-A8)/dt);
```

$$Left := \mathrm{D}_{1,1}(U)\left(x, t + \frac{1}{2}\,dt\right) + \frac{1}{24}\,dt^2\,\mathrm{D}_{1,1,1,1,1,1}(U)\left(x, t + \frac{1}{2}\,dt\right)$$

```
> B1 := series(U(x-dx, t+dt)-2*U(x, t+dt)+U(x+dx, t+dt), dx = 0);
```

$$B1 := \mathrm{D}_{1,1}(U)(x, t + dt)\,dx^2 + \frac{1}{12}\,\mathrm{D}_{1,1,1,1}(U)(x, t + dt)\,dx^4 + O(dx^6)$$

```
> B2 := convert(B1, polynom);
```

$$B2 := \mathrm{D}_{1,1}(U)(x, t + dt)\,dx^2 + \frac{1}{12}\,\mathrm{D}_{1,1,1,1}(U)(x, t + dt)\,dx^4$$

```
> B3 := convert(subs(h = dt, series(B2, dt = (1/2)*h, 4)), polynom);
```

$$B3 := \mathrm{D}_{1,1}(U)\left(x, t + \frac{1}{2}\,dt\right)\,dx^2 + \frac{1}{12}\,\mathrm{D}_{1,1,1,1}(U)\left(x, t + \frac{1}{2}\,dt\right)\,dx^4 + \frac{1}{2}\left(\frac{1}{12}\,\mathrm{D}_{1,1,1,1,2}(U)\left(x, t + \frac{1}{2}\,dt\right)\,dx^4 + \mathrm{D}_{1,1,2}(U)\left(x, t + \frac{1}{2}\,dt\right)\,dx^2\right)\,dt + \frac{1}{4}\left(\frac{1}{24}\,\mathrm{D}_{1,1,1,1,2,2}(U)\left(x, t + \frac{1}{2}\,dt\right)\,dx^4 + \frac{1}{2}\,\mathrm{D}_{1,1,2,2}(U)\left(x, t + \frac{1}{2}\,dt\right)\,dx^2\right)\,dt^2 + \frac{1}{8}\left(\frac{1}{72}\,\mathrm{D}_{1,1,1,1,2,2,2}(U)\left(x, t + \frac{1}{2}\,dt\right)\,dx^4 + \frac{1}{6}\,\mathrm{D}_{1,1,2,2,2}(U)\left(x, t + \frac{1}{2}\,dt\right)\,dx^2\right)\,dt^3$$

```
> B4 := subs(eq[12], subs({eq[10], eq[12]}, subs(eq[9], subs({eq[7], eq[9],
  eq[4], eq[8]}, B3))));
```

$$B4 := \mathrm{D}_{1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^2 + \frac{1}{12}\mathrm{D}_{1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^4 + \frac{1}{2}\left(\frac{1}{12}\mathrm{D}_{1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^4 + \mathrm{D}_{1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^2\right)dt + \frac{1}{4}\left(\frac{1}{24}\mathrm{D}_{1,1,1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^4 + \frac{1}{2}\mathrm{D}_{1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^2\right)dt^2 + \frac{1}{8}\left(\frac{1}{72}\mathrm{D}_{1,1,1,1,2,2,2}(U)\left(x, t+\frac{1}{2}dt\right)dx^4 + \frac{1}{6}\mathrm{D}_{1,1,2,2,2}(U)\left(x, t+\frac{1}{2}dt\right)dx^2\right)dt^3$$

```
> C1 := convert(series(U(x-dx, t)-2*U(x, t)+U(x+dx, t), dx = 0), polynom);
```

$$C1 := \mathrm{D}_{1,1}(U)(x, t)\,dx^2 + \frac{1}{12}\mathrm{D}_{1,1,1,1}(U)(x, t)\,dx^4$$

```
> C2 := convert(subs(h = t+(1/2)*dt, series(C1, t = h, 4)), polynom);
```

$$C2 := \mathrm{D}_{1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^2 + \frac{1}{12}\mathrm{D}_{1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^4 - \frac{1}{2}\left(\frac{1}{12}\mathrm{D}_{1,1,1,1,2}(U)\left(x, t+\frac{1}{2}dt\right)dx^4 + \mathrm{D}_{1,1,2}(U)\left(x, t+\frac{1}{2}dt\right)dx^2\right)dt + \frac{1}{4}\left(\frac{1}{24}\mathrm{D}_{1,1,1,1,2,2}(U)\left(x, t+\frac{1}{2}dt\right)dx^4 + \frac{1}{2}\mathrm{D}_{1,1,2,2}(U)\left(x, t+\frac{1}{2}dt\right)dx^2\right)dt^2 - \frac{1}{8}\left(\frac{1}{72}\mathrm{D}_{1,1,1,1,2,2,2}(U)\left(x, t+\frac{1}{2}dt\right)dx^4 + \frac{1}{6}\mathrm{D}_{1,1,2,2,2}(U)\left(x, t+\frac{1}{2}dt\right)dx^2\right)dt^3$$

```
> C3 := subs(eq[12], subs({eq[10], eq[12]}, subs(eq[9], subs({eq[7], eq[9],
  eq[4], eq[8]}, C2))));
```

$$C3 := \mathrm{D}_{1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^2 + \frac{1}{12}\mathrm{D}_{1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^4 - \frac{1}{2}\left(\frac{1}{12}\mathrm{D}_{1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^4 + \mathrm{D}_{1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^2\right)dt + \frac{1}{4}\left(\frac{1}{24}\mathrm{D}_{1,1,1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^4 + \frac{1}{2}\mathrm{D}_{1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^2\right)dt^2 - \frac{1}{8}\left(\frac{1}{72}\mathrm{D}_{1,1,1,1,2,2,2}(U)\left(x, t+\frac{1}{2}dt\right)dx^4 + \frac{1}{6}\mathrm{D}_{1,1,2,2,2}(U)\left(x, t+\frac{1}{2}dt\right)dx^2\right)$$

$$dt^3$$

```
> Right := simplify((theta*B4+(1-theta)*C3)/dx^2);
```

$$\begin{aligned}Right := {} & \frac{1}{12}\,\theta\,dt\,D_{1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^2 + \theta\,dt\,D_{1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right) \\ & + \frac{1}{288}\,\theta\,dt^3\,D_{1,1,1,1,2,2,2}(U)\left(x, t+\frac{1}{2}dt\right)dx^2 \\ & + \frac{1}{24}\,\theta\,dt^3\,D_{1,1,2,2,2}(U)\left(x, t+\frac{1}{2}dt\right) + D_{1,1}(U)\left(x, t+\frac{1}{2}dt\right) \\ & + \frac{1}{12}\,D_{1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^2 - \frac{1}{24}\,dt\,D_{1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^2 \\ & - \frac{1}{2}\,dt\,D_{1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right) + \frac{1}{96}\,dt^2\,D_{1,1,1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^2 \\ & + \frac{1}{8}\,dt^2\,D_{1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right) - \frac{1}{576}\,dt^3\,D_{1,1,1,1,2,2,2}(U)\left(x, t+\frac{1}{2}dt\right)dx^2 \\ & - \frac{1}{48}\,dt^3\,D_{1,1,2,2,2}(U)\left(x, t+\frac{1}{2}dt\right)\end{aligned}$$

```
> T := collect(simplify(Left-Right), dt);
```

$$\begin{aligned}T := {} & \left(-\frac{1}{288}\,\theta\,D_{1,1,1,1,2,2,2}(U)\left(x, t+\frac{1}{2}dt\right)dx^2 - \frac{1}{24}\,\theta\,D_{1,1,2,2,2}(U)\left(x, t+\frac{1}{2}dt\right)\right. \\ & \left. + \frac{1}{576}\,D_{1,1,1,1,2,2,2}(U)\left(x, t+\frac{1}{2}dt\right)dx^2 + \frac{1}{48}\,D_{1,1,2,2,2}(U)\left(x, t+\frac{1}{2}dt\right)\right)dt^3 \\ & + \left(-\frac{1}{12}\,D_{1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right) - \frac{1}{96}\,D_{1,1,1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^2\right) \\ & dt^2 + \left(-\theta\,D_{1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right) + \frac{1}{2}\,D_{1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)\right. \\ & \left. - \frac{1}{12}\,\theta\,D_{1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^2 + \frac{1}{24}\,D_{1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^2\right)dt \\ & - \frac{1}{12}\,D_{1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^2\end{aligned}$$

```
> # Crank Nicholson Method
  subs(theta = 1/2, T);
```

$$\left(-\frac{1}{12}\,D_{1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right) - \frac{1}{96}\,D_{1,1,1,1,1,1,1,1}(U)\left(x, t+\frac{1}{2}dt\right)dx^2\right)dt^2$$

$$-\frac{1}{12}\,\mathrm{D}_{1,1,1,1}(U)\left(x,\,t+\frac{1}{2}\,dt\right)\,dx^2$$

```
> # 4th order in space
  simplify(subs(theta = 1/2-dx^2/(12*dt), T));
```

$$\frac{1}{3456}\,dx^4\,dt^2\,\mathrm{D}_{1,1,1,1,2,2,2}(U)\left(x,\,t+\frac{1}{2}\,dt\right)+\frac{1}{288}\,dx^2\,dt^2\,\mathrm{D}_{1,1,2,2,2}(U)\left(x,\,t+\frac{1}{2}\,dt\right)$$
$$-\frac{1}{12}\,dt^2\,\mathrm{D}_{1,1,1,1,1,1}(U)\left(x,\,t+\frac{1}{2}\,dt\right)-\frac{1}{96}\,dt^2\,\mathrm{D}_{1,1,1,1,1,1,1,1}(U)\left(x,\,t+\frac{1}{2}\,dt\right)\,dx^2$$
$$+\frac{1}{144}\,\mathrm{D}_{1,1,1,1,1,1}(U)\left(x,\,t+\frac{1}{2}\,dt\right)\,dx^4$$

```
>
```