```
> restart;
> # Improved more Efficient Version
> with(LinearAlgebra):
> m:=5;
```

$$m := 5$$

```
> A:=Matrix([[0,0],[1/3,1/3]]);
```

$$A := \begin{bmatrix} 0 & 0 \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

```
> b:=Vector([1/4,3/4]);
```

$$b := \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$$

```
> c:=Vector([0,2/3]);
```

$$c := \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

```
> n:=Dimension(b);
```

$$n := 2$$

```
> onesvector:=Vector(n,1);
  fxi:=Vector(n);
  for j from 1 to n do
     fxi[j]:=f(t+c[j]*h,xi[j]);
  od;
  xirhs:=y(t)*onesvector+h*Multiply(A,fxi);
```

$$onesvector := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$fxi := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$fxi_1 := f(t, \xi_1)$$

$$fxi_2 := f\left(t + \frac{2}{3} h, \xi_2 \right)$$

$$xirhs := \begin{bmatrix} y(t) \\ y(t) + h\left(\frac{1}{3} f(t, \xi_1) + \frac{1}{3} f\left(t + \frac{2}{3} h, \xi_2 \right) \right) \end{bmatrix}$$

```
> xilhs:=Vector(n,i->xi[i]):
  for j from 1 to m-1
  do
      xilhs:=subs(xi=xilhs,xirhs);
      xilhs:=Map(x->series(x,h,j),xilhs);
      xilhs:=Map(x->convert(x,polynom),xilhs);
  od:
  xilhs;
```

$$\left[[y(t)], \left[y(t) + \frac{2}{3} f(t, y(t))\, h + \left(\frac{2}{9} D_1(f)(t, y(t)) + \frac{2}{9} D_2(f)(t, y(t))\, f(t, y(t)) \right) h^2 + \left(\frac{2}{27} D_{1,1}(f)(t, y(t)) + \frac{4}{27} D_{1,2}(f)(t, y(t))\, f(t, y(t)) + \frac{2}{27} D_{2,2}(f)(t, y(t))\, f(t, y(t))^2 + \frac{2}{27} D_2(f)(t, y(t))\, D_1(f)(t, y(t)) + \frac{2}{27} D_2(f)(t, y(t))^2\, f(t, y(t)) \right) h^3 \right] \right]$$

```
> R:=Vector(subs(xi=xilhs,fxi));
```

$$R := \left[[f(t, y(t))], \left[f\left(t + \frac{2}{3} h, y(t) + \frac{2}{3} f(t, y(t))\, h + \left(\frac{2}{9} D_1(f)(t, y(t)) + \frac{2}{9} D_2(f)(t, y(t))\, f(t, y(t)) \right) h^2 + \left(\frac{2}{27} D_{1,1}(f)(t, y(t)) + \frac{4}{27} D_{1,2}(f)(t, y(t))\, f(t, y(t)) + \frac{2}{27} D_{2,2}(f)(t, y(t))\, f(t, y(t))^2 + \frac{2}{27} D_2(f)(t, y(t))\, D_1(f)(t, y(t)) + \frac{2}{27} D_2(f)(t, y(t))^2\, f(t, y(t)) \right) h^3 \right) \right] \right]$$

```
> T:=y(t+h)-y(t)-h*Multiply(Transpose(b),R);
```

$$T := y(t + h) - y(t) - h\left(\frac{1}{4} f(t, y(t)) + \frac{3}{4} f\left(t + \frac{2}{3} h, y(t) + \frac{2}{3} f(t, y(t))\, h + \left(\frac{2}{9} D_1(f)(t, y(t)) + \frac{2}{9} D_2(f)(t, y(t))\, f(t, y(t)) \right) h^2 + \left(\frac{2}{27} D_{1,1}(f)(t, y(t)) + \frac{4}{27} D_{1,2}(f)(t, y(t))\, f(t, y(t)) + \frac{2}{27} D_{2,2}(f)(t, y(t))\, f(t, y(t))^2 + \frac{2}{27} D_2(f)(t, y(t))\, D_1(f)(t, y(t)) \right.\right.$$

$$+ \frac{2}{27} D_2(f)(t, y(t))^2 f(t, y(t)) \Bigg) h^3 \Bigg) \Bigg)$$

```
> S:=series(T,h,m);
```

$$S := (D(y)(t) - f(t, y(t))) h + \left(-\frac{1}{2} D_1(f)(t, y(t)) - \frac{1}{2} D_2(f)(t, y(t)) f(t, y(t)) + \frac{1}{2} D^{(2)}(y)(t) \right) h^2 +$$
$$\left(-\frac{1}{6} D_{1,1}(f)(t, y(t)) - \frac{1}{3} D_{1,2}(f)(t, y(t)) f(t, y(t)) - \frac{1}{6} D_{2,2}(f)(t, y(t)) f(t, y(t))^2 \right.$$
$$\left. - \frac{1}{6} D_2(f)(t, y(t)) D_1(f)(t, y(t)) - \frac{1}{6} D_2(f)(t, y(t))^2 f(t, y(t)) + \frac{1}{6} D^{(3)}(y)(t) \right) h^3 + \left(\frac{1}{24} D^{(4)}(y)(t) - \frac{1}{9} f(t, y(t)) D_{2,2}(f)(t, y(t)) D_1(f)(t, y(t)) - \frac{1}{18} D_2(f)(t, y(t))^2 D_1(f)(t, y(t)) \right.$$
$$- \frac{1}{6} D_{2,2}(f)(t, y(t)) D_2(f)(t, y(t)) f(t, y(t))^2 - \frac{1}{9} D_{1,1,2}(f)(t, y(t)) f(t, y(t))$$
$$- \frac{1}{18} D_2(f)(t, y(t))^3 f(t, y(t)) - \frac{2}{9} D_{1,2}(f)(t, y(t)) D_2(f)(t, y(t)) f(t, y(t))$$
$$- \frac{1}{9} D_{1,2}(f)(t, y(t)) D_1(f)(t, y(t)) - \frac{1}{9} D_{1,2,2}(f)(t, y(t)) f(t, y(t))^2$$
$$\left. - \frac{1}{27} f(t, y(t))^3 D_{2,2,2}(f)(t, y(t)) - \frac{1}{27} D_{1,1,1}(f)(t, y(t)) - \frac{1}{18} D_2(f)(t, y(t)) D_{1,1}(f)(t, y(t)) \right)$$
$$h^4 + O(h^5)$$

```
> eq[1]:=D(t->y(t))(t)=f(t,y(t));
```

$$eq_1 := D(y)(t) = f(t, y(t))$$

```
> for j from 1 to m-2
  do
      eq[j+1]:=simplify(subs(seq(eq[i],i=1..j),D(unapply(eq[j],t))(t)));
  od;
```

$$eq_2 := D^{(2)}(y)(t) = D_1(f)(t, y(t)) + D_2(f)(t, y(t)) f(t, y(t))$$

$$eq_3 := D^{(3)}(y)(t) = D_{1,1}(f)(t, y(t)) + 2 D_{1,2}(f)(t, y(t)) f(t, y(t)) + D_{2,2}(f)(t, y(t)) f(t, y(t))^2$$
$$+ D_2(f)(t, y(t)) D_1(f)(t, y(t)) + D_2(f)(t, y(t))^2 f(t, y(t))$$

$$eq_4 := D^{(4)}(y)(t) = D_{1,1,1}(f)(t, y(t)) + 3 D_{1,1,2}(f)(t, y(t)) f(t, y(t))$$
$$+ 3 D_{1,2,2}(f)(t, y(t)) f(t, y(t))^2 + 3 D_{1,2}(f)(t, y(t)) D_1(f)(t, y(t))$$
$$+ 5 D_{1,2}(f)(t, y(t)) D_2(f)(t, y(t)) f(t, y(t)) + f(t, y(t))^3 D_{2,2,2}(f)(t, y(t))$$
$$+ 3 f(t, y(t)) D_{2,2}(f)(t, y(t)) D_1(f)(t, y(t)) + 4 D_{2,2}(f)(t, y(t)) D_2(f)(t, y(t)) f(t, y(t))^2$$
$$+ D_2(f)(t, y(t)) D_{1,1}(f)(t, y(t)) + D_2(f)(t, y(t))^2 D_1(f)(t, y(t)) + D_2(f)(t, y(t))^3 f(t, y(t))$$

```
> simplify(subs(seq(eq[i],i=1..m-1),S));
```

$$\left(\frac{1}{216} D_{1,1,1}(f)(t, y(t)) + \frac{1}{72} D_{1,1,2}(f)(t, y(t))\, f(t, y(t)) + \frac{1}{72} D_{1,2,2}(f)(t, y(t))\, f(t, y(t))^2 \right.$$
$$+ \frac{1}{72} D_{1,2}(f)(t, y(t))\, D_1(f)(t, y(t)) - \frac{1}{72} D_{1,2}(f)(t, y(t))\, D_2(f)(t, y(t))\, f(t, y(t))$$
$$+ \frac{1}{216} f(t, y(t))^3\, D_{2,2,2}(f)(t, y(t)) + \frac{1}{72} f(t, y(t))\, D_{2,2}(f)(t, y(t))\, D_1(f)(t, y(t))$$
$$- \frac{1}{72} D_2(f)(t, y(t))\, D_{1,1}(f)(t, y(t)) - \frac{1}{72} D_2(f)(t, y(t))^2\, D_1(f)(t, y(t))$$
$$\left. - \frac{1}{72} D_2(f)(t, y(t))^3\, f(t, y(t)) \right)\, h^4 + O(h^5)$$