```
> restart;
> # Determine all two-stage IRK methods of order greater or equal 3
> with(LinearAlgebra):
> m:=4;
```

$$m := 4$$

```
> # Since order 3 then can check only for autonomous equation
  f:=(t,y)->g(y);
```

$$f := (t, y) \to g(y)$$

```
> A:=Matrix([[a11,a12],[a21,a22]]);
```

$$A := \begin{bmatrix} a11 & a12 \\ a21 & a22 \end{bmatrix}$$

```
> b:=Vector([b1,b2]);
```

$$b := \begin{bmatrix} b1 \\ b2 \end{bmatrix}$$

```
> c:=Vector([a11+a12,a21+a22]);
```

$$c := \begin{bmatrix} a11 + a12 \\ a21 + a22 \end{bmatrix}$$

```
> n:=Dimension(b);
```

$$n := 2$$

```
> onesvector:=Vector(n,1);
  fxi:=Vector(n):
  for j from 1 to n do
      fxi[j]:=f(t+c[j]*h,xi[j]);
  od:
  fxi;
  xirhs:=y(t)*onesvector+h*Multiply(A,fxi);
```

$$onesvector := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} g(\xi_1) \\ g(\xi_2) \end{bmatrix}$$

$$xirhs := \begin{bmatrix} y(t) + h\,(a11\,g(\xi_1) + a12\,g(\xi_2)) \\ y(t) + h\,(a21\,g(\xi_1) + a22\,g(\xi_2)) \end{bmatrix}$$

```
> T:=y(t+h)-y(t)-h*Multiply(Transpose(b),fxi);
  for j from 1 to m-1
  do
      T:=subs(xi=xirhs,T);
  od:
```

$$T := y(t+h) - y(t) - h\,(g(\xi_1)\,b1 + g(\xi_2)\,b2)$$

```
> T;
```

$$y(t+h) - y(t) - h\,(g(y(t) + h\,(a11\,g(y(t) + h\,(a11\,g(y(t) + h\,(a11\,g(\xi_1) + a12\,g(\xi_2))) + a12\,g(y(t) + h\,(a21\,g(\xi_1) + a22\,g(\xi_2))))) + a12\,g(y(t) + h\,(a21\,g(y(t) + h\,(a11\,g(\xi_1) + a12\,g(\xi_2))) + a22\,g(y(t) + h\,(a21\,g(\xi_1) + a22\,g(\xi_2)))))))\,b1 + g(y(t) + h\,(a21\,g(y(t) + h\,(a11\,g(y(t) + h\,(a11\,g(\xi_1) + a12\,g(\xi_2))) + a12\,g(y(t) + h\,(a21\,g(\xi_1) + a22\,g(\xi_2))))) + a22\,g(y(t) + h\,(a21\,g(y(t) + h\,(a11\,g(\xi_1) + a12\,g(\xi_2))) + a22\,g(y(t) + h\,(a21\,g(\xi_1) + a22\,g(\xi_2)))))))\,b2)$$

```
> S:=series(T,h,m);
```

$$S := (-g(y(t))\,b1 - g(y(t))\,b2 + \mathrm{D}(y)(t))\,h + \left(-\mathrm{D}(g)(y(t))\,g(y(t))\,(a21 + a22)\,b2 - \mathrm{D}(g)(y(t))\,g(y(t))\,(a11 + a12)\,b1 + \frac{1}{2}\,\mathrm{D}^{(2)}(y)(t)\right) h^2 + \left(-\left(a22\,\mathrm{D}(g)(y(t))^2\,g(y(t))\,a21 + a22^2\,\mathrm{D}(g)(y(t))^2\,g(y(t)) + a21\,\mathrm{D}(g)(y(t))^2\,g(y(t))\,a11 + a12\,\mathrm{D}(g)(y(t))^2\,g(y(t))\,a21 + \frac{1}{2}\,\mathrm{D}^{(2)}(g)(y(t))\,a21^2\,g(y(t))^2 + \mathrm{D}^{(2)}(g)(y(t))\,a21\,g(y(t))^2\,a22 + \frac{1}{2}\,\mathrm{D}^{(2)}(g)(y(t))\,a22^2\,g(y(t))^2\right)\,b2 - \left(a12\,\mathrm{D}(g)(y(t))^2\,g(y(t))\,a21 + a12\,\mathrm{D}(g)(y(t))^2\,g(y(t))\,a22 + a11^2\,\mathrm{D}(g)(y(t))^2\,g(y(t)) + a11\,\mathrm{D}(g)(y(t))^2\,g(y(t))\,a12 + \frac{1}{2}\,\mathrm{D}^{(2)}(g)(y(t))\,a11^2\,g(y(t))^2 + \mathrm{D}^{(2)}(g)(y(t))\,a11\,g(y(t))^2\,a12 + \frac{1}{2}\,\mathrm{D}^{(2)}(g)(y(t))\,a12^2\,g(y(t))^2\right)\,b1 + \frac{1}{6}\,\mathrm{D}^{(3)}(y)(t)\right) h^3 + O(h^4)$$

```
> eq[1]:=D(t->y(t))(t)=f(t,y(t));
```

$$eq_1 := \mathrm{D}(y)(t) = g(y(t))$$

```
> for j from 1 to m-2
  do
      eq[j+1]:=simplify(subs(seq(eq[i],i=1..j),D(unapply(eq[j],t))(t)));
  od;
```

$$eq_2 := \mathrm{D}^{(2)}(y)(t) = \mathrm{D}(g)(y(t))\, g(y(t))$$

$$eq_3 := \mathrm{D}^{(3)}(y)(t) = \mathrm{D}^{(2)}(g)(y(t))\, g(y(t))^2 + \mathrm{D}(g)(y(t))^2\, g(y(t))$$

```
> T:=simplify(subs(seq(eq[i],i=1..m-1),S));
```

$$T := (-g(y(t))\, b1 - g(y(t))\, b2 + g(y(t)))\, h + \left(-\mathrm{D}(g)(y(t))\, g(y(t))\, b2\, a21 \right.$$
$$- \mathrm{D}(g)(y(t))\, g(y(t))\, b2\, a22 - \mathrm{D}(g)(y(t))\, g(y(t))\, b1\, a11 - \mathrm{D}(g)(y(t))\, g(y(t))\, b1\, a12$$
$$\left. + \frac{1}{2}\, \mathrm{D}(g)(y(t))\, g(y(t)) \right)\, h^2 + \left(-b2\, a22\, \mathrm{D}(g)(y(t))^2\, g(y(t))\, a21 - b2\, a22^2\, \mathrm{D}(g)(y(t))^2\, g(y(t)) \right.$$
$$- b2\, a21\, \mathrm{D}(g)(y(t))^2\, g(y(t))\, a11 - b2\, a12\, \mathrm{D}(g)(y(t))^2\, g(y(t))\, a21$$
$$- \frac{1}{2}\, b2\, \mathrm{D}^{(2)}(g)(y(t))\, a21^2\, g(y(t))^2 - b2\, \mathrm{D}^{(2)}(g)(y(t))\, a21\, g(y(t))^2\, a22$$
$$- \frac{1}{2}\, b2\, \mathrm{D}^{(2)}(g)(y(t))\, a22^2\, g(y(t))^2 - b1\, a12\, \mathrm{D}(g)(y(t))^2\, g(y(t))\, a21$$
$$- b1\, a12\, \mathrm{D}(g)(y(t))^2\, g(y(t))\, a22 - b1\, a11^2\, \mathrm{D}(g)(y(t))^2\, g(y(t))$$
$$- b1\, a11\, \mathrm{D}(g)(y(t))^2\, g(y(t))\, a12 - \frac{1}{2}\, b1\, \mathrm{D}^{(2)}(g)(y(t))\, a11^2\, g(y(t))^2$$
$$- b1\, \mathrm{D}^{(2)}(g)(y(t))\, a11\, g(y(t))^2\, a12 - \frac{1}{2}\, b1\, \mathrm{D}^{(2)}(g)(y(t))\, a12^2\, g(y(t))^2$$
$$\left. + \frac{1}{6}\, \mathrm{D}^{(2)}(g)(y(t))\, g(y(t))^2 + \frac{1}{6}\, \mathrm{D}(g)(y(t))^2\, g(y(t)) \right)\, h^3 + O(h^4)$$

```
> T1:=coeff(T,h,1);
  eq1:=simplify(T1/g(y(t)))=0;
```

$$T1 := -g(y(t))\, b1 - g(y(t))\, b2 + g(y(t))$$

$$eq1 := -b1 - b2 + 1 = 0$$

```
> T2:=coeff(T,h,2);
  eq2:=simplify(T2/D(g)(y(t))/g(y(t)))=0;
```

$$T2 := -\mathrm{D}(g)(y(t))\, g(y(t))\, b2\, a21 - \mathrm{D}(g)(y(t))\, g(y(t))\, b2\, a22 - \mathrm{D}(g)(y(t))\, g(y(t))\, b1\, a11$$
$$- \mathrm{D}(g)(y(t))\, g(y(t))\, b1\, a12 + \frac{1}{2}\, \mathrm{D}(g)(y(t))\, g(y(t))$$

$$eq2 := -b2\, a21 - b2\, a22 - b1\, a11 - b1\, a12 + \frac{1}{2} = 0$$

```
> T3:=coeff(T,h,3);
```

$$T3 := -b2\, a22\, \mathrm{D}(g)(y(t))^2\, g(y(t))\, a21 - b2\, a22^2\, \mathrm{D}(g)(y(t))^2\, g(y(t))$$
$$- b2\, a21\, \mathrm{D}(g)(y(t))^2\, g(y(t))\, a11 - b2\, a12\, \mathrm{D}(g)(y(t))^2\, g(y(t))\, a21$$
$$- \frac{1}{2}\, b2\, \mathrm{D}^{(2)}(g)(y(t))\, a21^2\, g(y(t))^2 - b2\, \mathrm{D}^{(2)}(g)(y(t))\, a21\, g(y(t))^2\, a22$$

$$-\frac{1}{2}\,b2\,D^{(2)}(g)(y(t))\,a22^2\,g(y(t))^2 - b1\,a12\,D(g)(y(t))^2\,g(y(t))\,a21$$
$$- b1\,a12\,D(g)(y(t))^2\,g(y(t))\,a22 - b1\,a11^2\,D(g)(y(t))^2\,g(y(t))$$
$$- b1\,a11\,D(g)(y(t))^2\,g(y(t))\,a12 - \frac{1}{2}\,b1\,D^{(2)}(g)(y(t))\,a11^2\,g(y(t))^2$$
$$- b1\,D^{(2)}(g)(y(t))\,a11\,g(y(t))^2\,a12 - \frac{1}{2}\,b1\,D^{(2)}(g)(y(t))\,a12^2\,g(y(t))^2$$
$$+\frac{1}{6}\,D^{(2)}(g)(y(t))\,g(y(t))^2 + \frac{1}{6}\,D(g)(y(t))^2\,g(y(t))$$

```
> T3a:=coeff(T3,g(y(t)),1);
  eq3:=simplify(T3a/D(g)(y(t))^2)=0;
```

$$T3a := -b2\,a22\,D(g)(y(t))^2\,a21 - b2\,a22^2\,D(g)(y(t))^2 - b2\,a21\,D(g)(y(t))^2\,a11$$
$$- b2\,a12\,D(g)(y(t))^2\,a21 - b1\,a12\,D(g)(y(t))^2\,a21 - b1\,a12\,D(g)(y(t))^2\,a22$$
$$- b1\,a11^2\,D(g)(y(t))^2 - b1\,a11\,D(g)(y(t))^2\,a12 + \frac{1}{6}\,D(g)(y(t))^2$$

$$eq3 := -b2\,a22\,a21 - b2\,a22^2 - b2\,a21\,a11 - b2\,a12\,a21 - b1\,a12\,a21 - b1\,a12\,a22$$
$$- b1\,a11^2 - b1\,a11\,a12 + \frac{1}{6} = 0$$

```
> T3b:=coeff(T3,g(y(t)),2);
  eq4:=simplify(2*T3b/D(D(g))(y(t)))=0;
```

$$T3b := -\frac{1}{2}\,b2\,D^{(2)}(g)(y(t))\,a21^2 - b2\,D^{(2)}(g)(y(t))\,a21\,a22 - \frac{1}{2}\,b2\,D^{(2)}(g)(y(t))\,a22^2$$
$$- \frac{1}{2}\,b1\,D^{(2)}(g)(y(t))\,a11^2 - b1\,D^{(2)}(g)(y(t))\,a11\,a12 - \frac{1}{2}\,b1\,D^{(2)}(g)(y(t))\,a12^2$$
$$+\frac{1}{6}\,D^{(2)}(g)(y(t))$$

$$eq4 := -b2\,a21^2 - 2\,b2\,a22\,a21 - b2\,a22^2 - b1\,a11^2 - 2\,b1\,a11\,a12 - b1\,a12^2 + \frac{1}{3} = 0$$

```
> solve({eq1,eq2,eq3,eq4},{a11,a12,a21,a22,b1,b2});
```

$$\left\{ b2 = \frac{3\,(1 + 8\,a11\,a12 - 4\,a12 + 4\,a11^2 + 4\,a12^2 - 4\,a11)}{4\,(3\,a12^2 + 3\,a11^2 - 3\,a12 - 3\,a11 + 6\,a11\,a12 + 1)},\right.$$
$$a21 = -\frac{2\,a11 - 1 + a12}{3\,(1 + 8\,a11\,a12 - 4\,a12 + 4\,a11^2 + 4\,a12^2 - 4\,a11)},$$
$$\left. a22 = \frac{6\,a11^2 - 5\,a11 + 12\,a11\,a12 + 1 + 6\,a12^2 - 6\,a12}{3\,(2\,a11 + 2\,a12 - 1)^2},\right.$$

$$\left\{ b1 = \frac{1}{4\,(3\,a12^2 + 3\,a11^2 - 3\,a12 - 3\,a11 + 6\,a11\,a12 + 1)},\ a11 = a11,\ a12 = a12 \right\}$$