Math 466/666: Homework Assignment 1

This homework explores some properties of the Chebyshev polynomials related to optimal interpolation.

Students are encouraged to work together and consult resources outside of the required textbook for this assignment. Please cite any sources you consulted, including Wikipedia, other books, online discussion groups as well as personal communications. Be prepared to independently answer questions concerning the material on quizzes and exams.

Unless a disability makes it difficult, present all pencil-and-paper work in your own hand writing. To do this scan handwritten pages using a cell phone, document camera or flatbed scanner. Alternatively, you may write on a digital tablet with a writing stylus. If a computer was used to solve any part of a problem, include the code, input and output. Please upload your work as a single pdf file to WebCampus.

Equation (3.3–3) from the text states a theorem on the error in the approximations obtained using interpolating polynomials. When discussed in class we stated this result as

Theorem on Interpolating Polynomials: Given the distinct points x_i where $i = 1, \ldots, n$, let $p(x)$ be the unique interpolating polynomial of degree less than or equal $n-1$ such that

$$p(x_i) = f(x_i) \qquad \text{for} \qquad i = 1, \ldots, n.$$

Provided f has n derivatives, then for every t there is a corresponding ξ between $\min(t, x_1, \ldots, x_n)$ and $\max(t, x_1, \ldots, x_n)$ such that

$$f(t) = p(t) + \frac{q(t)}{n!} f^{(n)}(\xi) \qquad \text{where} \qquad q(t) = \textstyle\prod_{i=1}^{n} (t - x_i).$$

In this assignment we will consider how the choice of the points x_i affects $q(t)$ and the resulting bounds on the error in the approximation.

1. Consider the functions $g(t) = (t - c + \epsilon)(t - c - \epsilon)$ and $h(t) = (t - c)^2$. It is true or false that $g(t) < h(t)$ for all c, t and $\epsilon \neq 0$? If true explain why using mathematical reasoning, if false provide values of c, t and $\epsilon \neq 0$ such that $g(t) \geq h(t)$.

2. Consider the function

$$M(x_1, \ldots, x_n) = \max \left\{ \textstyle\prod_{i=1}^{n} |t - x_i| : t \in [-1, 1] \right\}.$$

For $n = 2$ find a choice for c_1 and c_2 such that

$$M(c_1, c_2) \leq M(x_1, x_2) \qquad \text{for all} \qquad x_1, x_2 \in \mathbf{R}.$$

In other words, find values c_1 and c_2 for x_1 and x_2 such that $M(x_1, x_2)$ is minimal.

3. Repeat the previous problem for $n = 3$ and $n = 4$. Please show all work and clearly explain your reasoning.

4. [Extra Credit and Math 666] Given arbitrary $n \in \mathbf{N}$, let the c_i be chosen such that

$$M(c_1, \dots, c_n) \le M(x_1, \dots, x_n) \qquad \text{for all} \qquad x_i \in \mathbf{R}.$$

Explain why the c_i must be distinct. Hint: Use your answer to the first question.

5. Use the angle addition formulas

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$
$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

and the Pythagorean theorem $\sin^2 a + \cos^2 a = 1$ to show $\cos(2a) = 2\cos^2 a - 1$.

6. Use techniques similar to those employed in the previous problem to express $\cos(3a)$ and $\cos(4a)$ as a function of $\cos a$.

7. [Extra Credit and Math 666] Given arbitrary $n \in \mathbf{N}$ with $n \ge 2$ use trigonometry to obtain the general reduction formula

$$\cos(na) = 2\cos a \cos\big((n-1)a\big) - \cos\big((n-2)a\big).$$

8. Define the Chebyshev polynomials as

$$T_n(t) = 2t\,T_{n-1}(t) - T_{n-2}(t) \qquad \text{where} \qquad T_0(t) = 1 \quad \text{and} \quad T_1(t) = t.$$

Find $T_2(t)$, $T_3(t)$ and $T_4(t)$.

9. Compare $T_n(t)$ to $\prod_{i=1}^{n}(t - c_i)$ for the values of c_i found earlier when $n = 2, 3, 4$. How are these functions related?

10. Compare $T_n(t)$ to the trigonometric identities for $\cos(na)$ when $n = 2, 3, 4$ by making the identification $t = \cos a$. How are these functions related?

11. The Chebyshev approximation theory implies that

$$c_i = \cos\left(\frac{\pi(i - \frac{1}{2})}{n}\right) \qquad \text{for} \qquad i = 1, \dots, n$$

leads to the minimal value of M such that

$$M(c_1, \dots, c_n) \le M(x_1, \dots, x_n) \qquad \text{for all} \qquad x_i \in \mathbf{R}.$$

Verify these values of c_i agree with those found in earlier for $n = 2, 3, 4$.

12. Given an arbitrary interval $[a, b]$ define

$$\widetilde{M}(x_1, \dots, x_n) = \max\left\{\, \textstyle\prod_{i=1}^{n} |t - x_i| : t \in [a, b] \,\right\}$$

and let $\widetilde{c}_i$ be chosen such that

$$\widetilde{M}(\widetilde{c}_1, \dots, \widetilde{c}_n) \le \widetilde{M}(x_1, \dots, x_n) \qquad \text{for all} \qquad x_i \in \mathbf{R}.$$

Find a relationship between $\widetilde{c}_i$ and the values of c_i defined earlier.