— Key —

Math/CS 466/666 Exam Part 1 Version A

1. State Taylor's theorem including all hypothesis and the remainder term.

Let $f:\mathbb{R}\to\mathbb{R}$ be a $n+1$ continuously differentiable function. Then there exists ξ between a and b such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^n}{n!}f^{(n)}(a) + R_n$$

where $R_n = \dfrac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$.

2. State Newton's method for solving $f(x) = 0$.

Suppose $f:\mathbb{R}\to\mathbb{R}$ is differentiable and let x_n be an approximation of the solution $f(x)=0$. Then Newton's method defines a sequence of approximation by $x_{n+1} = g(x_n)$ where $g(x) = x - \dfrac{f(x)}{f'(x)}$.

3. State the RK4 method for solving $y'(t) = f(t, y)$ with initial condition $y(t_0) = y_0$.

Define $t_n = t_0 + hn$ where $h>0$ and y_{n+1} by

$$y_{n+1} = y_n + (k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(t_n, y_n)$, $k_2 = hf(t_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

$k_3 = hf(t_n + \frac{h}{2}, y_n + \frac{k_2}{2})$, $k_4 = hf(t_n + h, y_n + k_3)$.

Then $y_n \approx y(t_n)$ where $y(t)$ is the solution to $y'(t) = f(t,y)$ with $y(t_0) = y_0$.

4. Use Taylor's theorem to show that the approximation

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

has order $\mathcal{O}(h^2)$ as $h \to 0$.

By Taylor's theorem

$$f(x+h) = \cancel{f(x)} + \cancel{hf'(x)} + \frac{h^2}{2} f''(x) + \cancel{\frac{h^3}{3!} f'''(x)} + \frac{h^4}{4!} f^{(4)}(\xi_1)$$

for some ξ_1 between x and $x+h$

and

$$f(x-h) = \cancel{f(x)} - \cancel{hf'(x)} + \frac{h^2}{2} f''(x) - \cancel{\frac{h^3}{3!} f'''(x)} + \frac{h^4}{4!} f^{(4)}(\xi_2)$$

for some ξ_2 between x and $x-h$,

It follows that

$$f(x+h) - 2\cancel{f(x)} + f(x-h) = h^2 f''(x) + \frac{h^4}{4!}\left(f^{(4)}(\xi_1) + f^{(4)}(\xi_2)\right)$$

Consequently, taking $M = \max\{ |f^{(4)}(t)| : t \in [x-\delta, x+\delta] \}$ for $\delta > 0$ yields

$$\left| \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - f''(x) \right| = \frac{h^4}{4!} \left| f^{(4)}(\xi_1) + f^{(4)}(\xi_2) \right|$$

$$\leq \frac{h^4}{4!} 2M = \frac{M}{12} h^4 \quad \text{for } |h| \leq \delta.$$

Therefore the approximation is $\mathcal{O}(h^2)$ as $h \to 0$.

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Consider the following table of divided differences

i	x_i	$f[x_i]$				
0	1.0	2.0				
			3.0			
1	2.0	5.0		$\boxed{}$		
			1.5		−0.500	
2	4.0	8.0		−2.50		0.160714
			−6.0		0.625	
3	5.0	2.0		1.25		
			−1.0			
4	8.0	−1.0				

5. Determine what value goes in the box.

$$\frac{1.5-3.0}{4-1} = \frac{-1.5}{3} = -0.5$$

6. Use the information in the table to write down the interpolating polynomial of degree 2 that passes through the points $(4, 8)$, $(5, 2)$ and $(8, -1)$.

$$p(x) = 8 - 6(x-4) + 1.25(x-4)(x-5)$$

alternatively going backwards

$$p(x) = -1 - (x-8) + 1.25(x-8)(x-5).$$

7. Given an approximation of $g(x)$ denoted by $\alpha(h,x)$ that is accurate to $\mathcal{O}(h^5)$ use Richardson's extrapolation to find an approximation $\beta(h,x)$ that is accurate to at least $\mathcal{O}(h^6)$ involving the terms $\alpha(h,x)$ and $\alpha(h/2,x)$.

$$\alpha(h,x) \approx g(x) + M h^5$$
$$\alpha\left(\tfrac{h}{2},x\right) \approx g(x) + M\left(\tfrac{h}{2}\right)^5 = g(x) + \frac{M}{2^5} h^4$$

$$2^5 \alpha\left(\tfrac{M}{2},x\right) \approx 2^5 g(x) + M h^4$$

Thus

$$2^5 \alpha\left(\tfrac{h}{2},x\right) - \alpha(h,x) \approx (2^5-1) g(x)$$

or

$$g(x) \approx \frac{2^5 \alpha\left(\frac{h}{2},x\right) - \alpha(h,x)}{2^5-1}$$

Consequently, take

$$\beta(h,x) = \frac{32}{31}\alpha\left(\tfrac{h}{2},x\right) - \frac{1}{31}\alpha(h,x).$$

8. Consider the row elimination and permutation matrices given by

$$E_{ij} = [r_i \leftarrow r_i - \alpha r_j] \qquad \text{and} \qquad P_{ij} = [r_i \leftrightarrow r_j].$$

Thus, E_{ij} is the matrix such that the product $E_{ij}A$ has the effect of subtracting α times row j of A from row i. Similarly P_{ij} is the matrix such that $P_{ij}A$ switches row i of A with row j. Find i, j, k, and ℓ such that $E_{4,1}P_{1,2} = P_{ij}E_{k\ell}$.

Switching rows 1 and 2, then eliminating the new row 1 from row 4 is the same as eliminating the old row 2 from row 4 and the switching rows 1 and 2. Therefore

$$E_{4,1}P_{1,2} = P_{1,2}E_{4,2}$$