Math 330 Exam 2 Review Sheet

1. Circle the correct answer for the following multiple choice questions.

(i) Find the orthogonal complement of $\mathcal{C}(A)$.
(A) $\mathcal{N}(A)$
(B) $\mathcal{C}(A)$
(C) $\mathcal{N}(A^T)$
(D) $\mathcal{C}(A^T)$
(E) none of these

(ii) Find the orthogonal complement of $\mathcal{C}(A^T)$.
(A) $\mathcal{N}(A)$
(B) $\mathcal{C}(A)$
(C) $\mathcal{N}(A^T)$
(D) $\mathcal{C}(A^T)$
(E) none of these

(iii) Find Q^TQ where Q is an orthogonal matrix.
(A) I
(B) 0
(C) A
(D) Q^{-1}
(E) none of these

(iv) Let Q be a matrix such that such that $\mathcal{C}(A) = \mathcal{C}(Q)$. Find the x that solves the least squares problem $Ax = b$.
(A) $(A^TQ)^{-1}A^Tb$
(B) $(A^TQ)^{-1}Q^Tb$
(C) $(Q^TA)^{-1}A^Tb$
(D) $(Q^TQ)^{-1}Q^Tb$
(E) none of these

2. Let

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Project the vector b onto the subspace given by the span of a.

3. Let Q be the matrix with orthonormal columns given by

$$Q = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 2\sqrt{2} & 1 \\ 2 & -\sqrt{2} & 2 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Find the x which minimizes $\|Qx - b\|$.

4. Find an orthonormal basis for the space spanned by the vectors

$$\begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

5. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Find $\det A$.

6. Let

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 4 & 2 \end{bmatrix}$$

Find $\det B$.

7. Let A and B be 4×4 matrices. Suppose $\det A = 2$ and $\det B = 3$.

(i) Find $\det(-A)$.

(ii) Find $\det(A^3 B)$.

(iii) Find $\det(B^T)$.

(iv) Find $\det(B^{-1})$.

(v) Find $\det(A + A)$.

8. Let

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Let B_j be the matrices formed according to Cramer's rule with the j-th column of A replaced by b. If

$$\det B_1 = 18, \qquad \det B_2 = -18, \qquad \text{and} \qquad \det B_3 = 36$$

then what is b? Find b_1, b_2 and b_3.

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Linear Algebra Formula Sheet

Orthogonal Projection

$$P = A(A^T A)^{-1} A^T$$

Least Squares

$$x = (A^T A)^{-1} A^T b$$

Gram-Schmidt

$$\begin{aligned}
\tilde{q}_1 &= a_1 & q_1 &= \tilde{q}_1 / \|\tilde{q}_1\| \\
\tilde{q}_2 &= a_2 - q_1(q_1 \cdot a_2) & q_2 &= \tilde{q}_2 / \|\tilde{q}_2\| \\
\tilde{q}_3 &= a_3 - q_1(q_1 \cdot a_3) - q_2(q_2 \cdot a_3) & q_3 &= \tilde{q}_3 / \|\tilde{q}_3\| \\
&\vdots & &\vdots \\
\tilde{q}_n &= a_n - q_1(q_1 \cdot a_n) - q_2(q_2 \cdot a_n) - \cdots - q_{n-1}(q_{n-1} \cdot a_n) & q_n &= \tilde{q}_n / \|\tilde{q}_n\|.
\end{aligned}$$

Least Squares using Gram-Schmidt

$$Rx = Q^T b \qquad \text{where} \qquad R = Q^T A$$

Combinatorial Determinant Formula

$$\det A = \sum_{\text{all } n \times n \text{ permuation matrices } P} (\det P)(\text{product of the diagonal of } PA)$$

Cofactor (Laplace's) Expansion

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \qquad \text{where} \qquad C_{ij} = (-1)^{i+j} \det M_{ij}$$

and M_{ij} is the submatrix of A resulting from deleting row i and column j.

Formula for the Inverse

$$A^{-1} = \frac{C^T}{\det(A)}$$

Cramer's Rule

$$x_1 = \frac{\det B_1}{\det A} \qquad x_2 = \frac{\det B_2}{\det A} \qquad \cdots \qquad x_n = \frac{\det B_n}{\det A}$$