Abel's Summation by Parts Formula

Given two sequences $(a_n)_{n\in\mathrm{N}}$ and $(b_n)_{n\in\mathrm{N}}$ of real numbers consider the partial sums

$$A_n = \sum_{k=1}^{n} a_k \qquad \text{and} \qquad B_n = \sum_{k=1}^{n} b_k.$$

Further define $A_0 = 0$ and $B_0 = 0$. Thus,

$$A_k - A_{k-1} = a_k \qquad \text{and} \qquad B_k - B_{k-1} = b_k \qquad \text{for} \qquad k = 1, 2, \ldots.$$

Now consider the difference $A_k B_k - A_{k-1} B_{k-1}$ of the products. Thus,

$$\begin{aligned} A_k B_k - A_{k-1} B_{k-1} &= A_k B_k - A_k B_{k-1} + A_k B_{k-1} - A_{k-1} B_{k-1} \\ &= A_k (B_k - B_{k-1}) + (A_k - A_{k-1}) B_{k-1} = A_k b_k + a_k B_{k-1}. \end{aligned}$$

Given $m, n \in \mathrm{N}$ with $m < n$, sum both sides from $k = m$ to n. Since the left side telescopes we have

$$\sum_{k=m}^{n} \big(A_k B_k - A_{k-1} B_{k-1} \big) = A_n B_n - A_{m-1} B_{m-1}.$$

Therefore

$$A_n B_n - A_{m-1} B_{m-1} = \sum_{k=m}^{n} A_k b_k + \sum_{k=m}^{n} a_k B_{k-1}.$$

This formula should be compared to the usual integration by parts formula. Let F and G be differentiable functions on $[a, b]$ with derivatives $F' = f$ and $G' = g$. Then by the product rule

$$(FG)' = FG' + F'G = Fg + fG.$$

Integrate both sides from a to b. The Fundamental Theorem of Calculus Part I implies that

$$\int_a^b (FG)' = F(b)G(b) - F(a)G(a).$$

Therefore,

$$F(b)G(b) - F(a)G(a) = \int_a^b F(x)g(x)\,dx + \int_a^b f(x)G(x)\,dx.$$

Comparing this formula to one above we see that sums play the role of integrals, a_k plays the role of the derivative of A_k, and similarly b_k is like the derivative of B_k.

Finally, the summation by parts formula in the book follows by taking $m = 1$ and using $A_0 = 0$ and $B_0 = 0$ to obtain

$$A_n B_n = \sum_{k=1}^{n} A_k b_k + \sum_{k=1}^{n} a_k B_{k-1}$$

and then identifying the books notation b_{n+1} with our B_n.