~ Key ~

Math 285 Quiz 1 Version A

1. State the order of the given ordinary differential equations and whether they are linear or non-linear.

(i) $\ddot{x} - \left(1 - \frac{\dot{x}^2}{3}\right)\dot{x} + x = 0$ Order $\boxed{2}$ Linearity $\boxed{\text{non-linear}}$

(ii) $y' + xy = \sin x$ Order $\boxed{1}$ Linearity $\boxed{\text{linear}}$

2. In 2005 David Lordkipanidze excavated a hominid skull in the Republic of Georgia that was dated to be 1.8 million years old. The half-life of carbon 14 is approximately 5700 years. What percentage of the original carbon 14 atoms are left in the skull?

$\dot{p} = -kp$

$p(t) = Ce^{-kt}$

$P(0) = C = 100$

$P(5700) = 100e^{-k5700} = 50$

$k = \frac{\ln 2}{5700}$

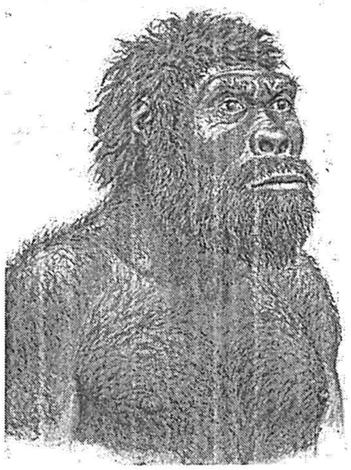
$P(1{,}800{,}000) = 100e^{-\frac{\ln 2}{5700}(1800000)}$

$\approx (100)2^{-18000/57}$

$\approx (100)2^{-315.8}$

$\approx 8.67 \times 10^{-94}$

≈ 0



An artist's rendition of what the original owner of the skull may have looked like. Credit: J.H. Matternes

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3. Consider the initial value problem

$$(*) \qquad \frac{dx}{dt} = f(x,t) \qquad \text{with} \qquad x(t_0) = x_0.$$

State the existence and uniqueness theorem which shows this ordinary differential equation has a unique solution on some open interval I containing t_0.

If $f(x,t)$ and $f_x(x,t)$ are continuous for $x \in (a,b)$ and $t \in (c,d)$, then for and $x_0 \in (a,b)$ and $t_0 \in (c,d)$ the initial value problem $(*)$ has a unique solution on some open interval I containing t_0.

4. Solve the following initial value problems:

(i) $\dot{x} = t$ with $x(0) = 3$.

$$x = \int t\,dt = \frac{1}{2}t^2 + C$$

$$x(0) = C = 3$$

$$x(t) = \frac{1}{2}t^2 + 3$$

(ii) $\dot{x} = te^t$ with $x(0) = 4$.

$$x = \int te^t dt = (A+Bt)e^t + C$$

Since $\frac{d}{dt}(A+Bt)e^t = (A+B+Bt)e^t$

then $A+B=0$ and $B=1$. Thus $A=-1$.

$$x = (-1+t)e^t + C$$

$$x(0) = -1 + C = 4, \quad C = 5$$

$$x(t) = (-1+t)e^t + 5$$

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5. Consider the autonomous first-order ordinary differential equation

$$\frac{dx}{dt} = x^2 - 3x$$

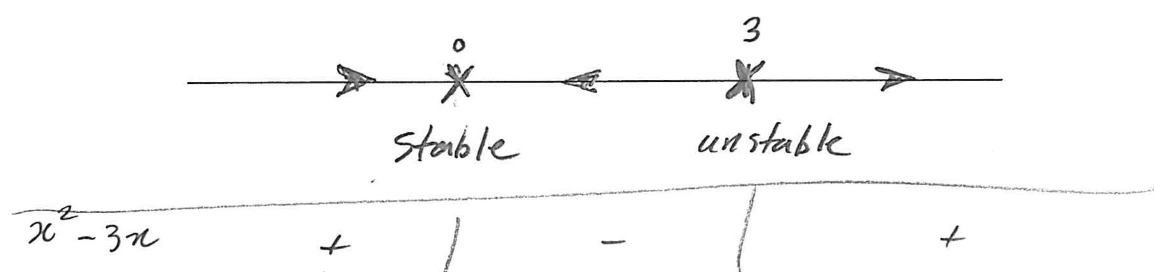
(i) Find the stationary points and determine their stability.

$x^2 - 3x = x(x-3)$

Stationary points are 0 and 3

Stability is indicated in the phase diagram below:

(ii) Draw a phase diagram. Label the stationary points with an cross $\times$ and draw arrows on the line below indicating the direction in which $x(t)$ is changing.



(iii) Find the unique solution to the differential equation that satisfies the initial condition $x(0) = 1$.

Partial fractions: $\frac{1}{x^2-3x} = \frac{A}{x} + \frac{B}{x-3}$

$A(x-3) + Bx = 1, \quad (A+B)x = 1+3A$

$A+B=0, \quad 1+3A=0, \quad A=-1/3, \quad B=1/3$

Separation of variables gives

$\int \frac{dx}{x^2-3x} = \int dt, \quad -\frac{1}{3}\ln|x| + \frac{1}{3}\ln|x-3| = t + C$

$-\frac{1}{3}\ln 1 + \frac{1}{3}\ln 2 = C, \quad C = \frac{1}{3}\ln 2$

Solution is $-\frac{1}{3}\ln x + \frac{1}{3}\ln(3-x) = t + \frac{1}{3}\ln 2$

simplification on back →

Problem 5(iii) continues...

$$\left(\frac{3-x}{2x}\right)^{1/3} = e^{t}$$

$$\frac{3}{2x} - \frac{1}{2} = e^{3t}$$

$$\frac{2x}{3} = \frac{1}{e^{3t} + \frac{1}{2}}$$

$$\boxed{x(t) = \frac{3}{2e^{3t} + 1}}$$

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6. Consider the differential equation $xy' - y = x^2 \sin x$.

(i) Find the general solution.

$y' - \frac{1}{x} y = x \sin x$

$I = e^{-\int \frac{1}{x} dx} = \frac{1}{x}$

$\frac{d}{dx}(y/x) = \sin x$

$y/x = \int \sin x \, dx = -\cos x + C$

$y(x) = -x \cos x + x C$

(ii) Find the unique solution that satisfies $y(1) = 1$.

$y(1) = -1 \cos 1 + C = 1$

$C = 1 + \cos 1$

$y(x) = -x \cos x + x(1 + \cos 1)$

7. Determine whether the following differential equations are exact.

(i) $2x - 1 + (3y + 7)y' = 0$

Compute $\frac{\partial}{\partial y}(2x-1) = 0$ and $\frac{\partial}{\partial x}(3y+7) = 0$

Since the above partial derivatives are equal the equation is exact.

(ii) $x^3 + y^3 + 3xy^2y' = 0$

Compute $\frac{\partial}{\partial y}(x^3+y^3) = 3y^2$ and $\frac{\partial}{\partial x}(3xy^2) = 3y^2$

Since ——————— " ———————

the equation is exact.

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8. Find the general solution to the homogeneous equation

$$x^2 + y^2 + (x^2 - xy)y' = 0.$$

$$y' = \frac{x^2+y^2}{x^2-xy} = \frac{1+(y/x)^2}{1-y/x}$$

Let $u = y/x$ then $y = ux$ and

$$y' = u + u'x = \frac{1+u^2}{1-u}$$

$$u'x = \frac{1+u^2}{1-u} - u = \frac{1+u^2-u+u^2}{1-n} = \frac{1-u+2n^2}{1-u}$$

$$\frac{1-u}{1-u-2u^2} = \frac{1-u}{(1-2u)(1+u)} = \frac{u-1}{(2u-1)(u+1)} = \frac{A}{2u-1} + \frac{B}{u+1}$$

see back →

9. Solve the Bernoulli initial value problem

$$y^{1/2}\frac{dy}{dx} + y^{3/2} = 1 \qquad \text{with} \qquad y(0) = 4.$$

$y' + y = y^{-1/2}$ so $n = -1/2$, $u = y^{1-n} = y^{3/2}$

$$u' = \frac{3}{2}y^{1/2}y' = \frac{3}{2}y^{1/2}(y^{-1/2} - y) = \frac{3}{2}(1 - y^{3/2}) = \frac{3}{2}(1-u)$$

Therefore $u' + \frac{3}{2}u = \frac{3}{2}$

$$I = e^{\int \frac{3}{2}dt} = e^{\frac{3}{2}t}$$

$$\frac{d}{dx}(ue^{\frac{3}{2}t}) = \frac{3}{2}e^{\frac{3}{2}t}$$

$$ue^{3/2t} = e^{3/2t} + C$$

$$u = 1 + Ce^{-3/2t}$$

$u(0) = 1 + C = 4$, $C = 3$

$$u(t) = 1 + 3e^{-3/2t}$$

8. continues...

$$A(u+1) + B(2u-1) = u-1$$

$$(A+2B)u + (A-B) = u-1$$

$$A+2B = 1, \quad A-B = -1$$

$$3B = 2, \quad B = 2/3$$

$$A = -1/3$$

Therefore

$$\int \frac{1-u}{1-u-2u^2}\,du = -\frac{1}{3}\ln|2u-1| + \frac{2}{3}\ln|u+1|$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Consequently

$$-\frac{1}{3}\ln|2u-1| + \frac{2}{3}\ln|u+1| = \ln|x| + C$$

$$\ln\frac{(u+1)^{2/3}}{(2u-1)^{1/3}} = \ln x + C$$

$$\frac{(u+1)^{2/3}}{(2u-1)^{1/3}} = C_2 x \quad \text{where } C_2 = e^C.$$

$$u^2+2u+1 = C_2^3 x^3 (2u-1)$$

$$u^2 + (2-2C_2^3x^3)u + 1 + C_2^3x^3 = 0$$

$$u = \frac{\cancel{2}(C_3x^3-1) \pm \sqrt{\cancel{4}(C_3x^3-1)^2 - 4(1+C_3x^3)}}{\cancel{2}}$$

$$u = C_3x^3 - 1 \pm \sqrt{C_3x^3(C_3x^3-3)}$$