Key

Math 285 Quiz 1 Version B

1. State the order of the given ordinary differential equations and whether they are linear or non-linear.

(i) $4xy' + 5y = \cos x$ Order $\boxed{1}$ Linearity $\boxed{\text{linear}}$

(ii) $\ddot{x} - (\cos x)\dot{x} + x = 0$ Order $\boxed{2}$ Linearity $\boxed{\text{non-linear}}$

2. Archaeologists used pieces of burned wood found at the site to date prehistoric paintings on the walls and ceilings of a cave in Lascaux, France. Determine the approximate age of the burned wood, if 14.5% of the carbon 14 found in living trees of the same type was still present. The half life of carbon 14 is 5700 years.

$\dot{p} = -kp$

$p(t) = Ce^{-kt}$

$p(0) = C = 100$

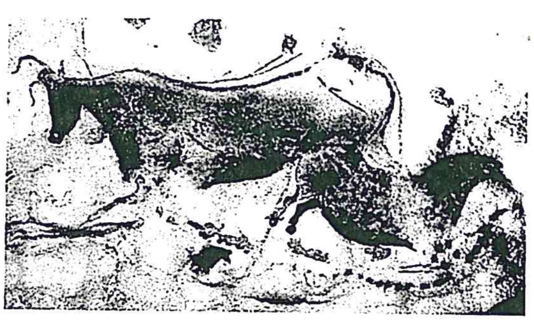
$p(5700) = 50 = 100e^{-k5700}$ $\quad k = \frac{\ln 2}{5700}$

$p(t) = 100e^{-\frac{\ln 2}{5700}t} = 14.5$

$$-\frac{\ln 2}{5700}t = \ln\frac{14.5}{100}$$

$$t = \frac{5700}{\ln 2}\cdot\ln\frac{100}{14.5} \approx \frac{5700}{\ln 2}\cdot 1.9310$$

$\approx 15,879.5$ years old.



Red Cow and First Chinese Horse.
Photograph: N. Aujoulat

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3. Consider the initial value problem

$$(*) \qquad \frac{dx}{dt} = f(x,t) \qquad \text{with} \qquad x(t_0) = x_0.$$

State the existence and uniqueness theorem which shows this ordinary differential equation has a unique solution on some open interval I containing t_0.

Suppose $x_0 \in (a,b)$ and $t_0 \in (c,d)$ and that f and f_x are continuous for all $x \in (a,b)$ and $t \in (c,d)$. Then $(*)$ has a unique solution on some open interval containing t_0.

4. Solve the following initial value problems:

(i) $\dot{x} = 3t^2$ with $x(0) = 5$.

$x = \int 3t^2 dt = t^3 + C$

$x(0) = C = 5$

$x(t) = t^3 + 5$

Trig identity:
$\cos^2 t = \frac{1+\cos 2t}{2}$

(ii) $\dot{x} = 2\cos^2 t$ with $x(0) = 1$.

$x = \int 2\cos^2 t \, dt = \int (1 + \cos 2t) dt$

$= t + \frac{1}{2}\sin 2t + C$

$x(0) = C = 1$

$x(t) = t + \frac{1}{2}\sin 2t + 1$

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5. Consider the autonoumous first-order ordinary differential equation

$$\frac{dx}{dt} = x^2 + x - 2$$

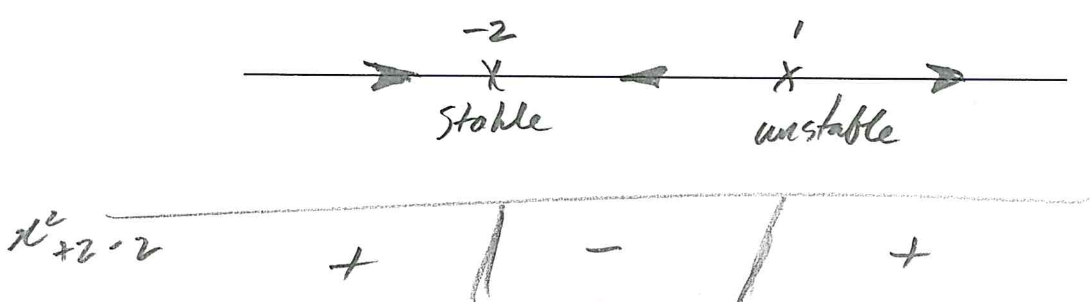
(i) Find the stationary points and determine their stability.

$x^2 + x - 2 = (x+2)(x-1)$

stationary points are $x = -2$ and $x = 1$

stability indicated on the phase diagram below;

(ii) Draw a phase diagram. Label the stationary points with an cross $\times$ and draw arrows on the line below indicating the direction in which $x(t)$ is changing.



(iii) Find the unique solution to the differential equation that satisfies the initial condition $x(0) = 1$.

Since $x = 1$ is a stationary point, the unique solution that passes through $x(0) = 1$ is $x(t) = 1$, the constant solution.

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6. Consider the differential equation $y' + y = e^{3x}$.

(i) Find the general solution. Integrating factor $\mu = e^{x}$

$$\frac{d}{dx}(ye^{x}) = e^{4x}$$

$$ye^{x} = \int e^{4x}dx = \frac{1}{4}e^{4x} + C$$

$$y = \frac{1}{4}e^{3x} + Ce^{-x}$$

(ii) Find the unique solution that satisfies $y(1) = 2$.

$$y(1) = \frac{1}{4}e^{3} + Ce^{-1} = 2, \quad C = e\left(2 - \frac{1}{4}e^{3}\right)$$

$$y(t) = \frac{1}{4}e^{3x} + \left(2 - \frac{1}{4}e^{3}\right)e^{1-x}$$

$$-8.213\, e^{-x}$$

7. Show the differential equation $2x - 1 + (3y + 7)y' = 0$ is exact and then find the general solution.

Since $\frac{\partial}{\partial y}(2x-1) = 0$ and $\frac{\partial}{\partial x}(3y+7) = 0$ are equal, then the equation is exact and there exists $F(x,y)$ such that $F_x = 2x-1$ and $F_y = 3y+7$.

$$F = \int(2x-1)dx = x^2 - x + C(y)$$

$$F_y = C'(y) = 3y + 7$$

$$C(y) = \int(3y+7)dy = \frac{3}{2}y^2 + 7y + C$$

Therefore $F(x,y) = x^2 - x + \frac{3}{2}y^2 + 7y + C$ and the general solution is

$$x^2 - x + \frac{3}{2}y^2 + 7y + C = 0.$$

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8. Find the general solution to the homogeneous equation

$$xyy' = 2x^2 + 3y^2.$$

$$y' = \frac{2x^2 + 3y^2}{xy} = 2\frac{x}{y} + 3\frac{y}{x}$$

Let $u = y/x$ then $y = ux$ and

$$y' = u'x + u = 2u^{-1} + 3u$$

$$xu' = 2u^{-1} + 2u$$

$$\int (2u^{-1} + 2u)^{-1}\,du = \int \frac{1}{x}\,dx$$

Since

$$\int (2u^{-1} + 2u)^{-1}\,du = \int \frac{u}{2+2u^2}\,du = \frac{1}{4}\ln(2+2u^2)$$

Then

$$\frac{1}{4}\ln(2+2u^2) = \ln x + C_1$$

$$\ln(2+2u^2) = \ln x^4 + C_2$$

$$2 + 2u^2 = C_3 x^4$$

Consequently, the general solution is

$$2x^2 + 2y^2 = C_3 x^6$$

or

$$y = \pm\sqrt{C_4 x^6 - x^2} = \pm x\sqrt{C_4 x^4 - 1}$$

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9. Solve the Bernoulli initial value problem

$$y' - 6xy = 2xy^2 \qquad \text{with} \qquad y(0) = 3.$$

$p(x) = -6x$, $q(x) = 2x$ and $n = 2$

Thus $u = y^{1-n} = 1/y$ and

$$u' = -1/y^2\, y' = -\frac{1}{y^2}(6xy + 2xy^2)$$

$$= -6\frac{x}{y} - 2x = -6xu - 2x$$

Hence

$$u' + 6xu = -2x$$

Integrating factor $\mu = e^{\int 6x\,dx} = e^{3x^2}$

$$\frac{d}{dx} u e^{3x^2} = -2xe^{3x^2}$$

$$ue^{3x^2} = -\int 2xe^{3x^2}dx = -\frac{1}{3}e^{3x^2} + C$$

Thus

$$u = -\frac{1}{3} + Ce^{-3x^2}$$

$$y = \frac{3}{C_2 e^{-3x^2} - 1}$$

$$y(0) = \frac{3}{C_2 - 1} = 3 \qquad \text{implies } C_2 = 2$$

Therefore

$$y(t) = \frac{3}{2e^{-3x^2} - 1}$$

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