Math 285 Quiz 2 Review on Euler's Method

Taylor's Theorem. *Let $f : \mathbf{R} \to \mathbf{R}$ be an $n+1$ times differentiable function. Then*

$$x(t+h) = x(t) + h\dot{x}(t) + \frac{h^2}{2!}\ddot{x}(t) + \cdots + \frac{h^n}{n!}x^{(n)}(t) + R_n$$

where

$$R_n = \int_t^{t+h} \frac{(t+h-s)^n}{n!} x^{(n+1)}(s)\,ds.$$

Proof. By the Fundamental Theorem of Calculus

$$x(t+h) - x(t) = \int_t^{t+h} \dot{x}(s)ds.$$

Integrating by parts where

$$u = \dot{x}(s), \quad dv = ds, \quad du = \ddot{x}(s)ds \quad \text{and} \quad v = -(t+h-s)$$

yields

$$\begin{aligned} x(t+h) - x(t) &= -(t+h-s)\dot{x}(s)\Big|_t^{t+h} + \int_t^{t+h} (t+h-s)\ddot{x}(s)ds \\ &= h\dot{x}(t) + \int_t^{t+h} (t+h-s)\ddot{x}(s)ds. \end{aligned}$$

Integrating again by parts where

$$u = \ddot{x}(s), \quad dv = (t+h-s)ds, \quad du = x^{(3)}(s)ds \quad \text{and} \quad v = -\tfrac{1}{2}(t+h-s)^2$$

yields

$$\begin{aligned} x(t+h) - x(t) &= h\dot{x}(t) - \frac{(t+h-s)^2}{2}\ddot{x}(s)\Big|_t^{t+h} + \int_t^{t+h} \frac{(t+h-s)^2}{2} x^{(3)}(s)ds \\ &= h\dot{x}'(t) + \frac{h^2}{2}\ddot{x}(t) + \int_t^{t+h} \frac{(t+h-s)^2}{2} x^{(3)}(s)ds. \end{aligned}$$

Repeated integration by parts finishes the proof. ///

Geometric Series Formula. *Let $A \in \mathbf{R}$. Then*

$$A^{n-1} + A^{n-2} + \cdots + A + 1 = \frac{A^n - 1}{A - 1}.$$

Proof. Let

$$S = A^{n-1} + A^{n-2} + \cdots + A + 1$$

so that

$$AS = A^n + A^{n-1} + \cdots + A^2 + A.$$

Now, subtracting yields

$$(1-A)S = 1 - A^n$$

from which the desired result follows. ///

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Limit of Compound Interest. *We have the following limit*

$$\left(1+\frac{r}{n}\right)^n \to e^r \qquad \textit{as} \qquad n\to\infty.$$

Proof. Since

$$\left(1+\frac{r}{n}\right)^n = \exp\left\{n\log\left(1+\frac{r}{n}\right)\right\}$$

and the exponential function is continuous, then it is enough to show that

$$n\log\left(1+\frac{r}{n}\right) \to r \qquad \text{as} \qquad n\to\infty.$$

Let $\theta = 1/n$. Then $\theta\to 0$ as $n\to\infty$. Consequently by L'Hôpital's rule

$$\lim_{n\to\infty} n\log\left(1+\frac{r}{n}\right) = \lim_{\theta\to 0}\frac{\log(1+r\theta)}{\theta} = \lim_{\theta\to 0}\frac{r/(1+r\theta)}{1} = r.$$

This finishes the proof. ///

Euler's Method. Euler's method for approximating the solution to the ordinary differential equation initial value problem

$$\dot{x} = f(x,t) \qquad \text{with} \qquad x(t_0) = x_0$$

is given as follows. Given $h>0$ define $t_k = t_0 + kh$ and

$$x_{k+1} = x_k + hf(x_k,t_k) \qquad \text{for} \qquad k = 0,1,2,\ldots.$$

Then x_k is an approximation of $x(t_k)$.

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Convergence of Euler's Method. Let $T > t_0$ and set $h = (T - t_0)/n$. Suppose $|\ddot{x}(t)| \le M$ and $|f_x(x,t)| \le L$ for all x and $t \in [t_0, T]$. Then $x_n \to x(T)$ as $h \to 0$.

Proof. Using Taylor's theorem

$$x_{k+1} - x(t_{k+1}) = x_k + hf(x_k, t_k) - \Big\{ x(t_k) + hf(x(t_k), t_k) + \int_t^{t+h} (t + h - s)\ddot{x}(s)ds \Big\}.$$

By the Fundamental Theorem of Calculus for $a < b$ we have

$$|f(b,t) - f(a,t)| = \Big| \int_a^b f_x(r,t)dr \Big| \le \int_a^b |f_x(r,t)|dr \le \int_a^b L dr = L|b-a|.$$

Therefore

$$|f(x_k, t_k) - f(x(t_k), t_k)| \le L|x_k - x(t_k)|$$

Similarly

$$\Big| \int_t^{t+h} (t+h-s)\ddot{x}(s)ds \Big| \le \int_t^{t+h} (t+h-s)M ds = -\frac{(t+h-s)^2}{2} M \Big|_t^{t+h} = \frac{h^2 M}{2}.$$

It follows that

$$\begin{aligned} |x_{k+1} - x(t_{k+1})| &\le |x_k - x(t_k)| + hL|x_k - x(t_k)| + \frac{h^2 M}{2} \\ &= A|x_k - x(t_k)| + \frac{h^2 M}{2} \qquad \text{where} \qquad A = 1 + hL. \end{aligned}$$

When $k = 0$ we obtain

$$|x_1 - x(t_1)| \le A|x_0 - x(t_0)| + \frac{h^2 M}{2} = \frac{h^2 M}{2} \qquad \text{since} \qquad x(t_0) = x_0.$$

When $k = 1$ we obtain

$$|x_2 - x(t_2)| \le A|x_1 - x(t_1)| + \frac{h^2 M}{2} \le A\Big\{\frac{h^2 M}{2}\Big\} + \frac{h^2 M}{2} = (A+1)\frac{h^2 M}{2}.$$

When $k = 2$ we obtain

$$|x_3 - x(t_3)| \le A|x_2 - x(t_2)| + \frac{h^2 M}{2} \le (A^2 + A + 1)\frac{h^2 M}{2}.$$

Continuing this pattern and applying the geometric series formula yields

$$|x_n - x(t_n)| \le (A^{n-1} + A^{n-2} + \cdots + A + 1)\frac{h^2 M}{2} = \frac{A^n - 1}{A - 1}\frac{h^2 M}{2}.$$

Therefore, using the limit of compound interest we have

$$\begin{aligned} |x_n - x(t_n)| &\le \frac{(1+hL)^n - 1}{(1+hL) - 1}\frac{h^2 M}{2} = \frac{M}{2L}\big((1+hL)^n - 1\big)h \\ &= \frac{M}{2L}\Big\{\Big(1 + \frac{T - t_0}{n}L\Big)^n - 1\Big\}h \\ &\to \frac{M}{2L}\big\{e^{(T-t_0)L} - 1\big\} \cdot 0 = 0 \qquad \text{as} \qquad h \to 0. \end{aligned}$$

This finishes the proof. ///