Math 285 Final Review Version A

1. Consider the initial value problem

$$\dot{x} = f(x,t) \qquad \text{with} \qquad x(t_0) = x_0.$$

(i) State the existence and uniqueness theorem showing this ordinary differential equation has a unique solution on some open interval I containing t_0.

If $f(x,t)$ and $f_x(x,t)$ are continuous for $x \in (a,b)$ and $t \in (c,d)$ then for any $x_0 \in (a,b)$ and $t_0 \in (c,d)$ the initial value problem

$$\dot{x} = f(x,t) \quad \text{with} \quad x(t_0) = x_0$$

has a unique solution on some open interval I containing t_0.

(ii) State the Runge–Kutta RK4 method for approximating this differential equation on the interval $[t_0, T]$.

Let $h = \frac{T-t_0}{n}$ and $t_k = t_0 + kh$ for $k = 0, \ldots, n$. The RK4 method for approximating $x(t_k)$ by x_k is given by

$$x_{k+1} = x_k + \frac{h}{6}(f_1 + 2f_2 + 2f_3 + f_4) \quad \text{for } k = 0, \ldots, n-1$$

where $f_1 = f(x_k, t_k)$, $f_2 = f(x_k + \frac{h}{2} f_1, t_k + \frac{h}{2})$

$f_3 = f(x_k + \frac{h}{2} f_2, t_k + \frac{h}{2})$ and $f_4 = f(x_k + h f_3, t_k + h)$.

(iii) State the definition of the Laplace transform $\mathcal{L}\{f\}$ of a function f.

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

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2. Solve the initial value problem $\dot{x} = \sin^2(3t)$ with $x(0) = 2$.

$$x = \int \sin^2(3t)\,dt = \int \frac{1-\cos 6t}{2}\,dt$$

$$x = \frac{t}{2} - \frac{1}{12}\sin 6t + C$$

$x(0) = C$ so $C = 2$

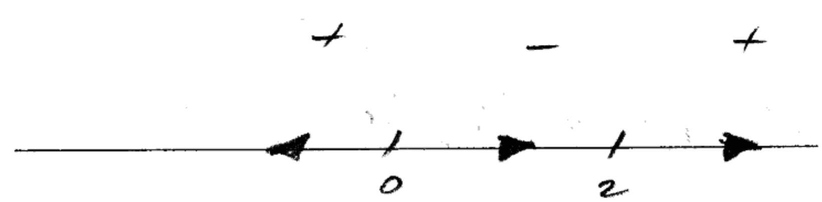
Unique solution is

$$x(t) = \frac{t}{2} - \frac{1}{12}\sin 6t + 2$$

3. Draw a phase diagram for the autonoumous first-order ordinary differential equation $\dot{x} = x^3 - 4x^2 + 4x$ on the line below. Label the stationary points with a cross $\times$ and draw arrows on the line indicating the direction in which $x(t)$ is changing.

$x^3 - 4x^2 + 4x = x(x^2 - 4x + 4) = x(x-2)^2$

Fixed points at $x = 0$ and $x = 2$.



4. Solve the initial value problem $\dot{x} - 2x = t$ with $x(0) = 1$.

$I = e^{-2t} \qquad \frac{d}{dt}(xe^{-2t}) = te^{-2t}$

$$xe^{-2t} = \int te^{-2t}\,dt = -\frac{1}{2}\int t\,de^{-2t} = -\frac{1}{2}\left(te^{-2t} - \int e^{-2t}\,dt\right)$$

$$= -\frac{t}{2}e^{-2t} + \frac{1}{2}\int e^{-2t}\,dt = -\frac{t}{2}e^{-2t} - \frac{1}{4}e^{-2t} + C$$

$x(t) = -\frac{t}{2} - \frac{1}{4} + Ce^{2t}$

$x(0) = -\frac{1}{4} + C = 1, \quad C = 5/4$

Unique soln:

$$x(t) = -\frac{t}{2} - \frac{1}{4} + \frac{5}{4}e^{2t}$$

5. Find the general solution to $\frac{dy}{dx} = \frac{x^2 + y^2}{x^2}$. $= 1 + (y/x)^2, \quad u = y/x$

$y = ux, \quad y' = u'x + u = 1 + v^2, \quad u'x = u^2 - u + 1$

$$\int \frac{dx}{x} = \int \frac{du}{u^2 - v + 1} = \int \frac{du}{(u - \frac{1}{2})^2 + 3/4} = \frac{4}{3}\int \frac{du}{4/3\,(u - \frac{1}{2})^2 + 1}$$

$$= \frac{4}{3}\int \frac{du}{[\frac{2}{\sqrt{3}}(u - \frac{1}{2})]^2 + 1} = \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}(u - \frac{1}{2})\right) + C$$

Therefore

$$\arctan\left(\frac{2}{\sqrt{3}}(y/x - 1/2)\right) = \frac{\sqrt{3}}{2}\ln|x| + C$$

$$\frac{2}{\sqrt{3}}(y/x - 1/2) = \tan\left(\frac{\sqrt{3}}{2}\ln|x| + C\right)$$

$$y = x\left[\frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}}{2}\ln|x| + C\right) + 1/2\right]$$

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6. Show that the ordinary differential equation

$$2xy - 9x^2 + (2y + x^2 + 1)y' = 0$$

is exact and find the general solution.

Since $\frac{\partial}{\partial y}(2xy - 9x^2) = 2x$ and $\frac{\partial}{\partial x}(2y + x^2 + 1) = 2x$ are the same, the the equation is exact. We now look for f such that $f_x = 2xy - 9x^2$ and $f_y = 2y + x^2 + 1$.

$f = \int(2xy - 9x^2)dx = x^2y - 3x^3 + C(y)$

$f_y = x^2 - 0 + C'(y) = 2y + x^2 + 1.$

$C'(y) = 2y+1 \quad C(y) = \int(2y+1)dy = y^2 + y + C$

Therefore the general solution is

$$x^2y - 3x^3 + y^2 + y = C.$$

7. Find the general solution to the differential equation

$$xy' - 2y = -x^3y^2.$$

This is Bernoulli equation with $n=2$. Then $u = y^{1-n} = 1/y$.

$u' = -1/y^2 \cdot y' = \frac{-1}{y^2}\left(\frac{2y}{x} - x^2y^2\right) = -\frac{2}{x}u + x^2$

We obtain $u' + \frac{2}{x}u = x^2$, $I = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

$\frac{d}{dx}(ux^2) = x^4$

$ux^2 = \int x^4 = \frac{1}{5}x^5 + C, \quad \frac{1}{y}x^2 = \frac{1}{5}x^5 + C$

The general solution

$$y(x) = \frac{x^2}{\frac{1}{5}x^5 + C}.$$

8. Consider the differential equation $\ddot{x} + 5\dot{x} + 6x = \sin 3t$.

(i) Find a particular solution for this differential equation.

$x_p = A\sin 3t + B\cos 3t$

$\dot{x}_p = 3A\cos 3t - 3B\sin 3t$

$\ddot{x}_p = -9A\sin 3t - 9B\cos 3t$

$6x_p = 6A\sin 3t + 6B\cos 3t$

$5\dot{x}_p = -15B\sin 3t + 15A\cos 3t$

$\ddot{x}_p = -9A\sin 3t - 9B\cos 3t$

$\sin 3t = (-3A - 15B)\sin 3t + (15A - 3B)\cos 3t$

$-3A - 15B = 1 \qquad 15A - 3B = 0 \qquad 5A - B = 0$

$\to -15A - 75B =$

$-78B = 5 \qquad B = \frac{-5}{78}$

$x_p(t) = \frac{-1}{78}\sin 3t - \frac{5}{78}\cos 3t \qquad A = \frac{B}{5} = \frac{-1}{78}$

(ii) Find the general solution to this differential equation.

$r^2 + 5r + 6 = (r+3)(r+2) \qquad r = -3, -2$

Complementary solution

$x_c(t) = C_1 e^{-3t} + C_2 e^{-2t}$

General solution

$x(t) = x_c(t) + x_p(t)$

$= C_1 e^{-3t} + C_2 e^{-2t} - \frac{1}{78}\sin 3t - \frac{5}{78}\cos 3t$

(iii) Find the unique solution such that $x(0) = 0$ and $\dot{x}(0) = 2$.

$x(0) = C_1 + C_2 - \frac{5}{78} = 0$

$\dot{x}(t) = -3C_1 e^{-3t} - 2C_2 e^{-2t} - \frac{3}{78}\cos 3t + \frac{15}{78}\sin 3t$

$\dot{x}(0) = -3C_1 - 2C_2 - \frac{3}{78} = 2 \qquad -C_1 = 2 + \frac{13}{78} = \frac{13}{6}$

$2C_1 + 2C_2 - \frac{10}{78} = 0$

$3C_1 + 3C_2 = -\frac{15}{78} = 0 \qquad C_2 = 2 + \frac{18}{78} = \frac{29}{13}$

Therefore the unique solution is

$x(t) = \frac{-13}{6} e^{-3t} + \frac{29}{13} e^{-2t} - \frac{1}{78}\sin 3t - \frac{5}{78}\cos 3t.$

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9. Consider the initial value problem

$$y'' - 2y' + 2 = e^{-t} \qquad \text{with} \qquad y(0) = 3, \quad y'(0) = -1.$$

Use Laplace transforms to solve for $Y(s) = \mathcal{L}\{y\}$. Do *not* invert to find y.

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{2\} = \mathcal{L}\{e^{-t}\}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 2[s\mathcal{L}\{y\} - y(0)] + \frac{2}{s} = \frac{1}{s+1}$$

$$s^2 Y(s) - 3s + 1 - 2sY(s) + 6 + \frac{2}{s} = \frac{1}{s+1}$$

$$(s^2 - 2s)Y(s) = 3s - 7 - \frac{2}{s} + \frac{1}{s+1}$$

$$Y(s) = \frac{3s - 7 - 2/s + 1/(s+1)}{s(s-2)}$$

10. Find the following inverse Laplace transforms:

(i) $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+s-2}\right\}(t) = u_2(t)\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s-1)}\right\}(t-2)$

$$\frac{1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

$$1 = (s-1)A + (s+2)B$$
$$0 = A + B$$
$$1 = -A + 2B, \quad B = 1/3, \quad A = -1/3$$

$$\ldots = u_2(t)\left[-\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t-2) + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t-2)\right]$$

$$= u_2(t)\left[-\frac{1}{3}e^{-2(t-2)} + \frac{1}{3}e^{t-2}\right] = \frac{1}{3}u_2(t)\left(e^{t-2} - e^{-2(t-2)}\right).$$

(ii) $\mathcal{L}^{-1}\left\{\frac{2s+1}{4s^2+4s+5}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{2s+1}{(2s+1)^2+4}\right\}(t)$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\}(t/2) = \frac{1}{2}e^{-t/2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}(t/2)$$

$$= \frac{1}{2}e^{-t/2}\cos[2(t/2)] = \frac{1}{2}e^{-t/2}\cos t.$$